OVERCONVERGENT MODULAR SYMBOLS

1. Modular symbols and L-values

1.1. **Introductory example.** Let f be the function on the upper half-plane defined by the q-expansion

$$f(z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2$$

with $q = e^{2\pi i z}$. Then $f \in S_2(\Gamma_0(11))$, i.e. f is a weight two cusp form for $\Gamma_0(11)$. We consider the period integrals

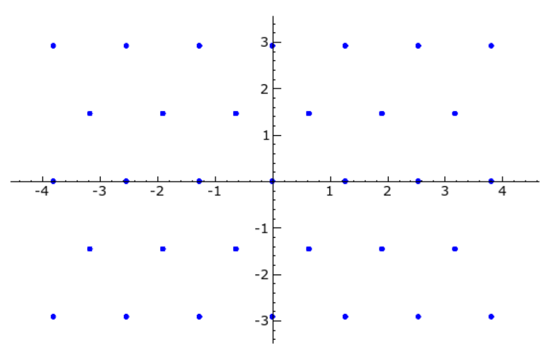
$$2\pi i \int_r^s f(z) dz$$

where r and s vary over $\mathbb{P}^1(\mathbb{Q})$. (These are path integrals in the upper-half plane along the semicircle connecting r to s.)

Let's do a little numerical experiment. Randomly choose a 100 pairs of r's and s in $\mathbb{Q}$, and compute the corresponding period integrals.[1]. Here are the first 10 period integrals:

$$\begin{array}{c}
1.269209304279553421688794617 00\ldots + 0.000000000000000000000000000000\ldots i \\
0.000000000000000000000000000000\ldots + 2.917633233876990458661779226 00\ldots i \\
0.634604652139776710844397308500\ldots + 1.458816616938495229330889613 00\ldots i \\
3.173023260698883554221986542 50\ldots + -1.458816616938495229330889613 00\ldots i \\
1.903813956419330132533191925 50\ldots + 1.458816616938495229330889613 00\ldots i \\
0.000000000000000000000000000000\ldots + 0.000000000000000000000000000000\ldots i \\
1.269209304279553421688794617 00\ldots + -2.917633233876990458661779226 00\ldots i \\
-1.903813956419330132533191925 50\ldots + 1.458816616938495229330889613 00\ldots i \\
-3.173023260698883554221986542 50\ldots + 1.458816616938495229330889613 00\ldots i \\
3.173023260698883554221986542 50\ldots + -1.458816616938495229330889613 00\ldots i
\end{array}$$

Plotting these points in the plane gives the following picture:



[1]Actually, in this computation, we actually only choose pairs of rational numbers which are $\Gamma_0(11)$-equivalent.

Not so random, eh? As you might guess, this collection of period integrals forms a lattice in $\mathbb{C}$. Moreover, one can explicitly write down generators of this lattice – namely, Ω_E^+ and $\frac{1}{2}\Omega_E^+ + \Omega_E^-$ where $\Omega_E^\pm$ are the Néron periods of the elliptic curve $X_0(11)$.

This period lattice is intimately related to the L-series of f. Namely, we have

$$2\pi i \int_{i\infty}^{0} f(z)dz = L(f,1) = \frac{1}{5}\cdot\Omega_E^+.$$

This first equality above is true much more generally.[2] The spirit of the second equality is true generally, but the presence of the factor of $\frac{1}{5}$ is very specific to this modular form, and in general, the exact value appearing is related to the Birch and Swinnerton-Dyer conjecture.[3]

More generally the period lattice contains the information of all twists of L-values. Namely, if χ is a Dirichlet character of conductor N, we have

$$L(f,\chi,1) = \frac{\tau(\chi)}{N} \sum_{a \bmod N} \overline{\chi}(a)\cdot 2\pi i \int_{i\infty}^{-\frac{a}{N}} f(z)dz \tag{3}$$

where $\tau(\chi)$ is the Gauss sum attached to χ.

This initial discussion is meant to convince you that these period integrals are quite interesting values. We now seek for an axiomatic (and algebraic!) way to describe them.

1.2. **Modular symbols.** Let Δ_0 equal the collection of degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$. (To connect to the previous discussion, think of the divisor $\{s\}-\{r\}$ as the path in the upper-half plane connecting r to s.) We then have a map ψ_f from Δ_0 to $\mathbb{C}$ defined by

$$\{s\}-\{r\} \mapsto 2\pi i \int_r^s f(z)dz.$$

Here, f is any cuspform of weight 2 on Γ a congruence subgroup. Of course, we have only defined ψ_f on elements of Δ_0 of the form $\{s\}-\{r\}$. But every element of Δ_0 is a sum of such elements, and so we extend ψ_f accordingly. We thus have constructed

$$\psi_f \in \mathrm{Hom}(\Delta_0, \mathbb{C})$$

where Hom here denotes additive maps.

The modularity of the function $f(z)$ tells us that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z)$$

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[2]Here is a heuristic argument for this equality:

$$2\pi i \int_{i\infty}^{0} f(z)dz = 2\pi i \int_{i\infty}^{0} \sum_n a_n e^{2\pi i n z} dz = 2\pi i \sum_n a_n \int_{i\infty}^{0} e^{2\pi i n z} dz \tag{1}$$

$$= \sum_n \frac{a_n}{n} e^{2\pi i n z}\Big|_{i\infty}^{0} dz = \sum_n \frac{a_n}{n} = L(f,1) \tag{2}$$

However, this argument suffers from serious convergence issues.

[3]The reason this period integral is not a $\mathbb{Z}$-multiple of Ω_E^+ is related to the fact that ∞ and 0 are not $\Gamma_0(11)$-equivalent.

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any matrix in Γ. In particular, the change of variables $u = \gamma z = \frac{az+b}{cz+d}$ yields following simple equality:

$$\int_{\gamma r}^{\gamma s} f(z)dz = \int_{\gamma r}^{\gamma s} f(\gamma z)(cz+d)^{-2}dz = \int_r^s f(z)dz. \tag{4}$$

We now give an algebraic description of this symmetry of period integrals. Namely, endow Δ_0 with the structure of a left $\mathrm{SL}_2(\mathbb{Z})$-module via linear fractional transformations and endow $\mathbb{C}$ with the trivial $\mathrm{SL}_2(\mathbb{Z})$ action. Then equation (4) converts into the fact that

$$\psi_f \in \mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$$

where the subscript of Γ indicates maps which are invariant under the action of $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$; that is, maps φ such that $\varphi(\gamma D) = \varphi(D)$ for all $\gamma \in \Gamma$ and all $D \in \Delta_0$. We call $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ the space of $\mathbb{C}$-valued modular symbols of level Γ.

1.3. **Back to $\Gamma_0(11)$.** Returning to the example of $f(z) \in S_2(\Gamma_0(11))$, we have a $\mathbb{C}$-valued modular symbol ψ_f of level $\Gamma := \Gamma_0(11)$ built out of the period integrals of f. Does the abstractly defined modular symbol space $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ contain any thing else other than multiples of ψ_f? Well, first let's point out that $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ is equipped with an involution ι given by

$$\iota(\varphi)(D) = \varphi(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) D)$$

since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ. So ι breaks $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ into plus and minus subspaces, and we get modular symbols ψ_f^+ and ψ_f^- such that $\psi_f = \psi_f^+ + \psi_f^-$. Moreover, the symbols $\psi_f^\pm$ takes values which are rational multiples of $\Omega_E^\pm$.

So is $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ bigger than 2-dimensional? Here's one way we can just write down explicit modular symbols. Let $\Delta = \mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ – i.e. we dropped the degree zero requirement. If we can define a Γ-invariant function on Δ, then by restriction we get an element of $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$.

What is a Γ-invariant function on Δ? Well, it's a function constant on Γ-orbits of Δ – i.e. a function on the cusps of $X_0(11)$! Since 11 is prime, there are only 2 cusps, 0 and ∞, and thus there are two linear independent such functions. However, the function which is constant on all cusps is killed after restriction. We will only get one more dimension of modular symbols this way. Namely, we could take

$$\varphi : \Delta \to \mathbb{C}$$

by

$$\varphi(r) = \begin{cases} 1 & \text{if } r \text{ is } \Gamma\text{-equivalent to } \infty \\ 0 & \text{if } r \text{ is } \Gamma\text{-equivalent to } 0, \end{cases}$$

and then restricting φ to Δ_0 then gives a (non-zero) element of $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$. (Note that flipping 0 and ∞ in the above definition just negates φ after restricting to Δ_0.)

Okay...so now we have that $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ is at least 3-dimensional. Any more? Well, we can again try to write $\varphi = \varphi^+ + \varphi^-$ as before. However, you should check that $\varphi^- = 0$, and this doesn't yield a new modular symbol.

I'm out of ideas for making new modular symbols. Can we prove that there are no more? Well, for starters, if we knew generators of Δ_0 as a $\mathbb{Z}[\Gamma]$-module, we would be in great shape. (Note that we just easily determined generators of Δ as a $\mathbb{Z}[\Gamma]$-module.)

For starters, let's at least write down a set of generators over $\mathbb{Z}$ (ignoring the Γ-action). Most naively, note that the set of $\{s\} - \{r\}$ as r and s vary over $\mathbb{P}^1(\mathbb{Q})$ generate over $\mathbb{Z}$. Even better, I claim that the set of $\{\frac{a}{b}\} - \{\frac{c}{d}\}$ for a, b, c, d satisfying $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ generate over $\mathbb{Z}$. Don't believe me? Well note that

$$\left\{\frac{29}{11}\right\} - \left\{\frac{0}{1}\right\} = \left\{\frac{29}{11}\right\} - \left\{\frac{-8}{-3}\right\} + \left\{\frac{8}{3}\right\} - \left\{\frac{5}{2}\right\} + \left\{\frac{5}{2}\right\} - \left\{\frac{-3}{-1}\right\} + \left\{\frac{3}{1}\right\} - \left\{\frac{2}{1}\right\} + \left\{\frac{2}{1}\right\} - \left\{\frac{-1}{0}\right\} + \left\{\frac{1}{0}\right\} - \left\{\frac{0}{1}\right\}$$

(Here $\frac{1}{0}$ just means ∞.) Indeed, this is Manin's continued fraction trick; the rational numbers appearing between $\frac{29}{11}$ and $\frac{0}{1}$ are just the convergents in the continued fraction expansion of $\frac{29}{11}$. This trick works generally yielding a $\mathbb{Z}$-generating set of Δ_0 indexed by $\mathrm{SL}_2(\mathbb{Z})$. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, write $[\alpha]$ for the divisor $\{\frac{b}{d}\} - \{\frac{a}{c}\}$.

Now to incorporate the action of Γ. A quick computation shows that if $\beta \in \mathrm{SL}_2(\mathbb{Z})$, then $[\beta\alpha] = \beta[\alpha]$. That is, multiplying on the left by β and then taking the associated divisor is the same as first taking the associated divisor and then acting with β by linear fractional transformations.

This simple formula tells us the following: if $\alpha_1, \cdots, \alpha_d$ are a system of right coset representations for $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$, then $[\alpha_1], \cdots, [\alpha_d]$ are $\mathbb{Z}[\Gamma]$-generators of Δ_0. Indeed, if $[\beta]$ is one of our $\mathbb{Z}$-generators of Δ_0, write $\beta = \gamma\alpha_i$ for some i and some $\gamma \in \Gamma$. Then $[\beta] = \gamma \cdot [\alpha_i]$. In particular, a modular symbol is uniquely determined by its values on the finite list of divisors $[\alpha_1], \cdots, [\alpha_d]$.

OK, let's apply these ideas to our particular case of $\Gamma = \Gamma_0(11)$. In this example, we would need 12 coset representations as $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 12$. Hmph. That's a large number to start with. Let's take a simpler example, say $\Gamma_0(2)$. In this case, the index is 3 and the following are right coset representatives:

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

(Note that two matrices represent the same right coset in $\Gamma_0(2)$ if their bottom rows, thought of as elements as $\mathbb{P}^1(\mathbb{F}_2)$, agree.) Their associated divisors are:

$$\{0\} - \{\infty\}, \{-1\} - \{0\}, \{\infty\} - \{-1\}. \tag{5}$$

Are these linearly independent over $\mathbb{Z}[\Gamma_0(2)]$? Nope. Not even over $\mathbb{Z}$ as their sum is 0. Any other relations? Well, note that

$$\{-1\} - \{0\} = -(\{0\} - \{-1\}) = -(\{\gamma(-1)\} - \{\gamma(0)\}) = -\gamma(\{-1\} - \{0\}).$$

where $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \in \Gamma_0(2)$.

Now, let's say we have φ some $\mathbb{C}$-valued modular symbol of level $\Gamma_0(2)$. Then φ is certainly determined by its values on the three divisors in (5). Moreover, by $\Gamma_0(2)$-invariance, we have

$$\varphi(\{-1\} - \{0\}) = -\varphi(\gamma(\{-1\} - \{0\})) = -\varphi(\{-1\} - \{0\}),$$

and thus φ vanishes on $\{-1\} - \{0\}$. Then, since our three divisors sum to 0, we get

$$0 = \varphi(\{0\} - \{\infty\}) + \varphi(\{\infty\} - \{-1\}).$$

Therefore, the value of φ on $\{\infty\} - \{0\}$ determines the value of φ on $\{-1\} - \{\infty\}$, and thus on all divisors. In particular, this space of modular symbols is at most 1-dimensional. The same trick of writing down a symbol on Δ and restricting again works, and we see that this space is exactly 1-dimensional. (Note that there are no cuspforms of weight 2 and level $\Gamma_0(2)$.)

How to generalize this to the $\Gamma_0(11)$ case? Well, let me tell you where I got the three divisors in (5). First note that the ideal triangle connecting ∞ to 0 to -1 is a fundamental domain for $\Gamma_0(2)$. So it's the "boundary" of this domain that gives rise to our 3 divisors. The fact that the three divisors sum to 0 comes from going around the triangle. The relation involving $\{0\} - \{-1\}$ arises from the fact that the bottom edge of the triangle is identified with itself modulo $\Gamma_0(2)$ – i.e. from the gluing data describing how to wrap the fundamental domain to get $X_0(2)$.

So for $X_0(11)$ – make a nice fundamental domain and see what happens!

1.4. **Higher weight case.** Now let's take an arbitrary cusp form f in $S_k(\Gamma)$ with Γ a congruence subgroup. The relevant period integrals attached to f are of the form

$$\int_r^s f(z)z^j dz$$

where j ranges between 0 and $k-2$. We should thus beef up the associated modular symbol to encode all of these periods, and we do so by changing the space where the symbols take values.

Namely, let $V_g(\mathbb{C}) = \mathrm{Sym}^g(\mathbb{C}^2)$ realized as the space of homogeneous polynomials of degree g in $\mathbb{C}[X, Y]$. Moreover, we endow this space with a right action (seriously, a right action) of $\mathrm{SL}_2(\mathbb{Z})$ by setting

$$(P\big|\gamma)(X, Y) = P\left((X, Y) \cdot \gamma^*\right) = P(dX - cY, -bX + aY)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\gamma^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and $P \in V_g(\mathbb{C})$. In fact, this action makes sense for any matrix with non-zero determinant.

We then define

$$\psi_f(\{s\} - \{r\}) = 2\pi i \int_r^s f(z)(zX + Y)^{k-2} dz \in V_{k-2}(\mathbb{C}).$$

This gives an element of

$$\mathrm{Hom}(\Delta_0, V_{k-2}(\mathbb{C}))$$

and as before we exhibit some Γ-invariant property. Namely, for any $\gamma \in M_2(\mathbb{Z})$ with non-zero determinant, define a right action on $\mathrm{Hom}(\Delta_0, V_{k-2}(\mathbb{C}))$ via

$$(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma.$$

A simple computation, which again uses the modularity of f, yields that

$$\psi_f \in \mathrm{Hom}_\Gamma(\Delta_0, V_{k-2}(\mathbb{C}))$$

where the subscript Γ denotes the subspace of maps which are invariant under the above action of Γ. Explicitly, these are the maps such that

$$\varphi(\gamma D) = \varphi(D)|\gamma^{-1}$$

for $\gamma \in \Gamma$ and $D \in \Delta_0$.

1.5. **Modular symbols in general.** So far we've considered modular symbols with values in $\mathbb{C}$ (with trivial action) and with values in $\mathrm{Sym}^g(\mathbb{C}^2)$. Let's write down the general theory here as laid out in [1]. To this end, let $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ denote the set of degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$. Then Δ_0 (the Steinberg module) has the structure of a left $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Q})]$-module where $\mathrm{GL}_2(\mathbb{Q})$ acts via linear fractional transformations.

Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let V be a right $\mathbb{Z}[\Gamma]$-module. We endow the set of additive homomorphisms $\mathrm{Hom}(\Delta_0, V)$ with the structure of a right Γ-module by defining

$$(\varphi\big|\gamma)(D) := \varphi(\gamma D)\big|\gamma$$

for $\varphi : \Delta_0 \to V$, $D \in \Delta_0$ and $\gamma \in \Gamma$.

For φ in $\mathrm{Hom}(\Delta_0, V)$, we say that φ is a *V-valued modular symbol* on Γ if $\varphi\big|\gamma = \varphi$ for all $\gamma \in \Gamma$; we denote the space of all V-valued modular symbols by $\mathrm{Symb}_\Gamma(V)$. Thus, for an additive homomorphism $\varphi : \Delta_0 \to V$,

$$\varphi \in \mathrm{Symb}_\Gamma(V) \iff \varphi(\gamma D) = \varphi(D)\big|\gamma^{-1} \text{ for all } \gamma \in \Gamma \text{ and } D \in \Delta_0.$$

We remark that if $\mathcal{H}$ denotes the upper-half plane and $\widetilde{V}$ is the associated locally constant sheaf of V on $\mathcal{H}/\Gamma$, then there is a canonical isomorphism

$$\mathrm{Symb}_\Gamma(V) \cong H^1_c(\mathcal{H}/\Gamma, \widetilde{V}),$$

provided that the order of any torsion element of Γ acts invertibly on V (see [1, Prop 4.2]). In this course, we however focus on the explicit description of modular symbols given by maps rather than by cohomology classes.

The modules V considered in this course will have the addition structure of a right action by $S_0(p)$ where

$$S_0(p) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \text{ such that } (a,p) = 1,\ p|c \text{ and } ad - bc \neq 0 \right\}.$$

Given this additional structure, one can define a Hecke-action on $\mathrm{Symb}_\Gamma(V)$; if ℓ is a prime, then the Hecke operator T_ℓ is given by the double coset $\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right) \Gamma$. For example, if $\Gamma = \Gamma_0(N)$ and $\ell \nmid N$, then

$$\varphi|T_\ell = \varphi| \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{a=0}^{l-1} \varphi| \left(\begin{smallmatrix} 1 & a \\ 0 & \ell \end{smallmatrix}\right).$$

If $q|N$, we write U_q for T_q, and we have

$$\varphi|U_q = \sum_{a=0}^{q-1} \varphi| \left(\begin{smallmatrix} 1 & a \\ 0 & q \end{smallmatrix}\right).$$

We further remark that when $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ, this matrix acts as an involution on $\mathrm{Symb}_\Gamma(V)$. When 2 acts invertibly on V, we then have a natural decomposition

$$\mathrm{Symb}_\Gamma(V) \cong \mathrm{Symb}_\Gamma(V)^+ \oplus \mathrm{Symb}_\Gamma(V)^-$$

into ± 1-eigenspaces for this action.

This last chunk of text was just cut and pasted from [4]. But now that we have Hecke operators defined in general, go back and figure out how those symbols we wrote down for $\Gamma_0(11)$ behave under Hecke!

1.6. ***L*-values.** As mentioned before, the modular symbol ψ_f should know special values of the L-series of f. Generalizing equation (1), we have the following relation between L-values and period integrals:

$$2\pi i \int_{i\infty}^{0} f(z) z^j dz = \frac{j!}{(-2\pi i)^j} L(f, j+1)$$

for $0 \leq j \leq k-2$.

If we set $\sum_{j=0}^{k-2} c_j X^j Y^{k-2-j} \in \mathrm{Sym}^{k-2}(\mathbb{C}^2)$ equal to the value of ψ_f at $\{\infty\}-\{0\}$, we can then relate the coefficient c_j to the $L(f, j+1)$. Namely, we have

$$\begin{aligned}\psi_f(\{\infty\}-\{0\}) &= 2\pi i \int_{i\infty}^{0} f(z)(zX+Y)^{k-2}dz \\ &= 2\pi i \int_{i\infty}^{0} f(z) \sum_{j=0}^{k-2} \binom{k-2}{j} z^j X^j Y^{k-2-j} dz \\ &= \sum_{j=0}^{k-2} \binom{k-2}{j} \left(2\pi i \int_{i\infty}^{0} f(z) z^j dz\right) X^j Y^{k-2-j} \\ &= \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{j!}{(-2\pi i)^j} L(f, j+1) X^j Y^{k-2-j},\end{aligned}$$

which implies that

$$c_j = \binom{k-2}{j} \frac{j!}{(-2\pi i)^j} L(f, j+1).$$

Note that this matches our previous formula when $j = 0$.

1.7. **Eichler-Shimura.** We close by stating a theorem of Eichler and Shimura which relates these spaces of modular symbols to modular forms. Namely:

Theorem 1.1. *There is an isomorphism*

$$\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})) \cong M_k(\Gamma, \mathbb{C}) \oplus S_k(\Gamma, \mathbb{C})$$

which respects the action of Hecke on both sides.

You may ask, why are there two copies of the cusp forms appearing? That's just because of the action of ι that we observed at the very start. Attached to each cusp form f, there are two modular symbols ψ_f^+ and ψ_f^-. You may ask, why are Eisenstein series appearing? Well, look back to the maps we originally wrote down on Δ and then restricted to Δ_0. These will account for Eisenstein series as you probably already noted if you did the exercises at the end of section 1.5.

1.8. **p-adic L-functions.** As it will be relevant in the last lecture, we mention now a connection to p-adic L-functions. Namely, one can construct the p-adic L-function of f out of the modular symbol ψ_f.

Before doing this, a few words on what p-adic L-functions are, at least in the case of weight 2. Their job is to interpolate special values of the L-series of f. Namely, they should "know" the values $L(f, \chi, 1)$ where χ runs over all Dirichlet characters of conductor p^n. On the surface, this makes no sense because these are complex numbers and p-adic L-functions should live in the p-adic world. To remedy this, consider the modular symbol $\psi_f^\pm$ all of whose values are algebraic multiples of some fixed complex number $\Omega_f^\pm$ as we noted for the $\Gamma_0(11)$ example.[4] Set $\varphi_f^\pm = \frac{\psi_f^\pm}{\Omega_f^\pm}$ which is a modular symbol taking values in $\overline{\mathbb{Q}}$. Also, set $\varphi_f = \varphi_f^+ + \varphi_f^-$.

[4] We are side stepping the appropriate normalization of this period which would be needed to get μ-invariants correct.

From equation (3), it now follows that the L-values $\dfrac{L(f,\chi,1)}{\Omega_f^{\pm}}$ are all algebraic. (Here the sign of the period depends on whether χ is even or odd.) Since these L-values are algebraic, we can and do consider them as either complex numbers or p-adic numbers.

So what kind of p-adic gadget can encode all of these twisted L-values. Well, in the spirit of Tate's thesis, one thinks of the association:

$$\chi \mapsto \frac{L(f,\chi,1)}{\Omega_f^{\pm}},$$

and then thinks of L-functions as functions on character spaces. In the p-adic world, the relevant character space is $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p)$. Note that every Dirichlet character of p-power conductor is in this space. So the p-adic L-function should be able to take as an input any $\mathbb{C}_p$-valued character on $\mathbb{Z}_p^\times$ and return a p-adic number. Even better, it will be able to take as an input any "nice enough" function on $\mathbb{Z}_p^\times$, and return a p-adic number. That is, the p-adic L-function will be a distribution – i.e something in the dual of a space of nice p-adic functions.

All of this will be made more precise in the next lecture, but for now, we want to build a gadget which takes in nice functions on $\mathbb{Z}_p^\times$ and spits out numbers. Moreover, when you input a finite-order character on $\mathbb{Z}_p^\times$, it spits out the relevant twisted L-value. To accomplish this, it suffices to write down a measure on $\mathbb{Z}_p^\times$ which we now do (at least in the case of ordinary weight two modular forms). That is, let α be the unique unit root of $x^2 - a_p x + p$ and define

$$\mu_f(a + p^n\mathbb{Z}_p) = \frac{1}{\alpha^n}\varphi_f(\{\infty\} - \{a/p^n\}) - \frac{1}{\alpha^{n+1}}\varphi_f(\{\infty\} - \{a/p^{n-1}\}).$$

With this definition, the p-adic L-function μ_f satisfies the following interpolation property: for χ a non-trivial finite order character of conductor p^n, we have

$$\int_{\mathbb{Z}_p^\times} \chi d\mu_f = \frac{1}{\alpha^n}\tau(\chi)\frac{L(f,\overline{\chi},1)}{\Omega_f^{\pm}}.$$

Many questions should arise: where does the formula defining μ_f come from? Is it even additive: i.e. if one takes an open of the form $a + p^n\mathbb{Z}_p$ and writes it as a disjoint union each of the form $b + p^{n+1}\mathbb{Z}_p$, is that formula compatible with such a union? Lastly, how does it connect to L-values? We will attempt to answer all of these questions in that last lecture. For now, let me just say, "you'll see", "yes" and "equation (3)", and point out that the p-adic L-function of f is being built out of the data of the modular symbol attached to f evaluated at infinitely many different divisors.

2. Distributions leading to overconvergent modular symbols

2.1. **A brief story about p-adic families.** In the mid 80s, Hida constructed p-adic families of ordinary modular forms. The constructions and methods of this theory rely crucially on the following observation: the dimension of the subspace of ordinary modular forms in $S_k(\Gamma)$ is independent of the weight k. This allowed Hida to p-adically interpolate the finite-dimensional spaces of ordinary forms as the weight varied.

This phenomenon of course cannot occur for arbitrary modular forms as the dimension of the full space $S_k(\Gamma)$ heavily depends upon k (growing unboundedly as

k increases). Thus, one can't hope for a nice (finite, flat) family over weight space interpolating $S_k(\Gamma)$.

To circumvent this problem, Coleman had the ingenious idea to pass to a much larger space, namely, $M_k^\dagger(\Gamma)$, the space of overconvergent modular forms. This is an infinite-dimensional Banach space (on which U_p acts completely continuously). Moreover, it contains $M_k(\Gamma)$, the space of classical forms. And, doubly moreover, Coleman proves that an overconvergent eigenform is classical if the p-adic valuation of its U_p-eigenvalue is strictly less than $k-1$.[5]

Now at least there is a hope that the Banach spaces $D_k^\dagger(\Gamma)$ can be put into a nice family as they all at least have the same dimension, namely infinity! And indeed this is exactly what Coleman does, and he succeeds in making p-adic families of modular forms (although, unlike Hida theory, there are forms in these families of classical weight which are not themselves classical).

In this course, we will describe Steven's analogue of overconvergent modular forms (called overconvergent modular symbols) along with his analogue of Coleman's control theorem. To this end, we are going to replace the spaces $V_g(\mathbb{C})$ with p-adic spaces whose dimensions don't move around with g. In particular, we will write down spaces $\mathcal{D}_g(\mathbb{Z}_p)$ of p-adic distributions (which are infinite-dimensional Banach spaces). Moreover, these spaces will admit surjective maps from $\mathcal{D}_g(\mathbb{Z}_p)$ to $V_g(\mathbb{Q}_p)$. We then replace the space $\mathrm{Symb}_\Gamma(V_g(\mathbb{Q}_p))$ with the space $\mathrm{Symb}_\Gamma(\mathcal{D}_g(\mathbb{Z}_p))$ which will be the space of overconvergent modular symbols. And indeed these will be the spaces that ultimately vary well p-adically.

2.2. **Distributions.** We start by defining the simplest of the distribution spaces which we will need for this course. Namely, let $\mathbf{A}$ denote the collection of power series with coefficients in $\mathbb{Q}_p$ which converge on the unit disc of $\mathbb{C}_p$. That is:

$$\mathbf{A} = \{f(z) \in \mathbb{Q}_p[[z]] \ : \ f(z) = \sum_n a_n z^n \text{ and } |a_n| \to 0 \text{ as } n \to \infty\}.$$

Note that $\mathbf{A}$ is a Banach space under the norm:

$$||f|| = \max_n |a_n|$$

where $f(z) = \sum_n a_n z^n$. We then define our space of distributions $\mathbf{D}$ by

$$\mathbf{D} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{A}, \mathbb{Q}_p).$$

Note that $\mathbf{D}$ is a Banach space under the norm

$$||\mu|| = \sup_{\substack{f \in \mathbf{A} \\ f \neq 0}} \frac{|\mu(f)|}{||f||}.$$

2.3. **Moments of distributions.** This distribution space is quite concrete. Indeed, $\mathbf{D}$ can be identified with the space of bounded sequences in $\mathbb{Q}_p$ endowed with the sup norm. To get this identification, we just note that the $\mathbb{Q}_p$-span of all monomials $\{z^j\}_{j=0}^\infty$ is dense in $\mathbf{A}$ (check this!). Thus, a distribution $\mu \in \mathbf{D}$ is uniquely

[5] A classical form of level Γ cannot have a U_p-eigenvalue with p-adic valuation larger than $k-1$. So it's only forms of slope $k-1$ that are not completely explained by this theorem. These are called the *critical slope* forms.

determined by its values on these monomials. In particular, we get an injective map

$$\mathbf{D} \xrightarrow{M} \prod_{j=0}^{\infty} \mathbb{Q}_p.$$

$$\mu \mapsto \{\mu(z^j)\}_j$$

We call $\{\mu(z^j)\}_j$ the sequence of moments attached to μ.

We claim that the image of M is exactly the collection of bounded sequences. Indeed, if $\{\alpha_n\}$ is a bounded sequence of elements in $\mathbb{Q}_p$, we can just define a distribution $\mu \in \mathbf{D}$ by simply setting

$$\mu(z^n) = \alpha_n.$$

Then μ extends linearly to a functional on $\mathbf{A}$ by setting

$$\mu(\sum a_n z^n)) = \sum a_n \alpha_n,$$

which converges as $|a_n| \to 0$ and $\{\alpha_n\}$ is bounded. We leave it for you to check that for any $\mu \in \mathbf{D}$, it's associated sequence of moments is a bounded sequence.

The upshot of this subsection is the following: although $\mathbf{D}$ has the complicated definition of the dual of convergent power series on the closed unit disc of $\mathbb{C}_p$, it has the very concrete realization as the collection of bounded sequences in $\mathbb{Q}_p$.

2.4. **The action of $\Sigma_0(p)$.** Let

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \text{ such that } p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}$$

For each non-negative integer k, we define a weight k action of $\Sigma_0(p)$ on $\mathbf{A}$ by

$$(\gamma \cdot_k f)(z) = (a + cz)^k \cdot f\left(\frac{b + dz}{a + cz}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and $f \in \mathbf{A}$. Then $\Sigma_0(p)$ acts on $\mathbf{D}$ on the right by

$$(\mu\big|_k \gamma)(f) = \mu(\gamma \cdot_k f)$$

where $\mu \in \mathbf{D}$. When we view $\mathbf{A}$ or $\mathbf{D}$ as a $\Sigma_0(p)$-module endowed with a weight k action, we write $\mathbf{A}_k$ or $\mathbf{D}_k$.

Note that by "transport of structure" we have also defined a $\Sigma_0(p)$-action on the space of bounded sequences in $\mathbb{Q}_p$ via our identification from the last section. However, don't expect anything special; this action is just a mess in the language of sequences.

2.5. **Finite-dimensional quotients.** For an integer $k \geq 0$, consider

$$V_k := V_k(\mathbb{Q}_p) := \mathrm{Sym}^k(\mathbb{Q}_p^2),$$

the space of homogeneous polynomials of degree k in X and Y with coefficients in $\mathbb{Q}_p$. We recall that we endow the space $V_k(\mathbb{Q}_p)$ with the structure of a right $\mathrm{GL}_2(\mathbb{Q}_p)$-module by

$$(P\big|\gamma)(X, Y) = P\left((X, Y) \cdot \gamma^*\right) = P(dX - cY, -bX + aY)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $P \in V_k(\mathbb{Q}_p)$.

There is an $\Sigma_0(p)$-equivariant map

$$\rho_k : \mathbf{D}_k \to V_k(\mathbb{Q}_p)$$
$$\mu \mapsto \int (Y - zX)^k \, d\mu(z)$$

where the integration takes place coefficient by coefficient. That is

$$\rho_k(\mu) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu(z^j) X^j Y^{k-j} \in V_k(\mathbb{Q}_p).$$

(Check that this map is really $\Sigma_0(p)$-invariant!)

2.6. **Specialization.** Let $\Gamma_0 = \Gamma_0(Np)$. We refer to $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)$ as a space of weight k overconvergent modular symbols.[6] These spaces of overconvergent modular symbols naturally map to the space of classical modular symbols. Indeed, the map $\rho_k : \mathbf{D}_k \to V_k$ induces a map

$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k) \to \mathrm{Symb}_{\Gamma_0}(V_k)$$

which we refer to as the *specialization map*. Note that ρ_k^* is Hecke-equivariant as ρ_k is $\Sigma_0(p)$-equivariant.[7]

Now the source of the specialization map is infinite-dimensional while the target is finite-dimensional, and so the kernel is certainly huge. Indeed, U_p acts on the target with slope at most $k+1$.[8] Thus, the entire subspace of the source on which U_p acts with slope larger than $k+1$ must be in the kernel. The following control theorem of Stevens says that apart from the critical slope cases (i.e. slope exactly equal to $k+1$), this is precisely what happens. Namely,

Theorem 2.1 (Stevens)**.** *We have*

$$\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)^{(<k+1)} \longrightarrow \mathrm{Symb}_{\Gamma_0}(V_k)^{(<k+1)}$$

is an isomorphism. That is, the specialization map restricted to the subspace where U_p acts with slope strictly less than $k+1$ is an isomorphism.

This should be viewed as analogous to Coleman's theorem on small slope forms being classical.[9] Indeed, this theorem says that a classical eigensymbol of small enough slope lifts uniquely to an overconvergent eigensymbol. Note that in Coleman's world, classical forms are a sub of overconvergent forms. However, in this setting, classical modular symbols are a quotient of overconvergent modular symbols. A proof of Theorem 2.1 will be sketched in the next lecture.[10]

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[6]Note that $\mathbf{D}_k$ is equipped with a natural Γ_0-action, but this action does not extend to Γ.

[7]Recall that $V_k = \mathrm{Sym}^k(\mathbb{Q}_p^2)$ and so modular symbols in $\mathrm{Symb}_{\Gamma_0}(V_k)$ correspond to modular forms of weight $k+2$, and not weight k.

[8]Here slope is the p-adic valuation of an eigenvalue of U_p.

[9]In comparing the two theorems remember the shift between k and $k+2$.

[10]What happens in the critical slope case is more subtle. See [5] and [2].

2.7. **The truth about our distribution spaces.** To be more honest, the space $\mathbf{D}$ is not really the space we are ultimately interested in (although it will be the space we primarily work with). Indeed, p-adic L-functions actually live in a smaller space of distributions. Distributions in $\mathbf{D}$ can only be evaluated on functions which are expressible as a convergent power series on the entire closed unit disc. We want to be evaluating on functions on $\mathbb{Z}_p$. Thinking of $\mathbb{Z}_p$ as inside the closed unit ball, a random continuous function will not extend to a rigid analytic function on disc. For instance, any non-constant but locally constant functions (*e.g.* a character) cannot be represented by a single power series.

The functions on $\mathbb{Z}_p$ we will be considering are the "locally analytic" ones; that is, $f : \mathbb{Z}_p \to \mathbb{C}_p$ is locally analytic if for every point y of $\mathbb{Z}_p$, the function f is expressible as a power series on some ball around y in $\mathbb{Z}_p$. (Since $\mathbb{Z}_p$ is compact, one only needs finitely many power series to represent f on $\mathbb{Z}_p$.) Let $\mathcal{A}$ denote the collection of locally analytic functions on $\mathbb{Z}_p$. Note that finite-order characters on $\mathbb{Z}_p^\times$ (considered as functions on $\mathbb{Z}_p$ by extending by 0) are locally analytic since they are locally constant. One can also check that any character on $\mathbb{Z}_p^\times$ (not necessarily finite-order) is also locally analytic.

We want to define the space of "locally analytic" distributions to be the (continuous) dual of $\mathcal{A}$. However, we haven't given a topology yet to $\mathcal{A}$. This is a little bit tricky because $\mathcal{A}$ is not a Banach space.

To proceed, for each $r \in |\mathbb{C}_p^\times|$, we set

$$B[\mathbb{Z}_p, r] = \{z \in \mathbb{C}_p \mid \text{there exists some } a \in \mathbb{Z}_p \text{ with } |z-a| \leq r\}.$$

For example, if $r \geq 1$ then $B[\mathbb{Z}_p, r]$ is the closed disc in $\mathbb{C}_p$ of radius r around 0. If $r = \frac{1}{p}$ then $B[\mathbb{Z}_p, r]$ is the disjoint union of the p discs of radius $\frac{1}{p}$ around the points $0, 1, \ldots, p-1$.

Let $\mathbf{A}[r]$ denote the collection of $\mathbb{Q}_p$-rigid analytic functions on $B[\mathbb{Z}_p, r]$. For example, if $r \geq 1$

$$\mathbf{A}[r] = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[[z]] \text{ such that } \{|a_n| \cdot r^n\} \to 0 \right\}.$$

In particular, $\mathbf{A}[1]$ is nothing other than the space $\mathbf{A}$ we've been studying this section. If $r = \frac{1}{p}$, then an element of $\mathbf{A}[r]$ is a function on $B[\mathbb{Z}_p, r]$ which when restricted to each of the p discs of radius $\frac{1}{p}$ is representable by a convergent power series with coefficients in $\mathbb{Q}_p$. The norm on $\mathbf{A}[r]$ is given by the supremum norm. That is, if $f \in \mathbf{A}[r]$ then

$$||f||_r = \sup_{z \in B[\mathbb{Z}_p, r]} |f(z)|_p.$$

For $r_1 > r_2$, there is a natural restriction map $\mathbf{A}[r_1] \to \mathbf{A}[r_2]$ which is injective. Since $\mathbb{Z}_p$ is contained in $B[\mathbb{Z}_p, r]$ for any $r > 0$, there is a natural restriction map $\mathbf{A}[r] \to \mathcal{A}$ which is also injective. Moreover, since each element in $\mathcal{A}$ is representable by finitely many power series, any such element is in the image of $\mathbf{A}[r]$ for some r. Thus,

$$\mathcal{A} = \varinjlim_{r} \mathbf{A}[r]$$

as r tends to 0 in the limit. We thus endow $\mathcal{A}$ with the inductive limit topology which is the strongest topology making all of the inclusions $\mathbf{A}[r] \to \mathcal{A}$ continuous.

Lastly, we set $\mathcal{D}$ equal to the continuous $\mathbb{Q}_p$-dual of $\mathcal{A}$, which we will call the space of locally analytic distributions on $\mathbb{Z}_p$.

Since we have a map $\mathbf{A} \to \mathcal{A}$ by restriction to $\mathbb{Z}_p$, dualizing gives an injective map $\mathcal{D} \to \mathbf{D}$. The injectivity of this map follows from:

Fact: The span of the monomials $\{z^j\}$ is dense in $\mathcal{A}$.

Thus, we can and do identify $\mathcal{D}$ as a subspace of $\mathbf{D}$; that is, every locally analytic distribution is in fact a rigid analytic distribution.

We note that in a completely analogous way, we can endow $\mathcal{A}$ and $\mathcal{D}$ with a weight k action by $\Sigma_0(p)$, and again we write $\mathcal{A}_k$ and $\mathcal{D}_k$. It is also true that the same formula defining ρ_k also gives a $\Sigma_0(p)$-map from $\mathcal{D}_k$ to $V_k(\mathbb{Q}_p)$. Lastly, Steven's control theorem will hold true if we replace $\mathbf{D}_k$ with $\mathcal{D}_k$. Indeed, for any $h \in \mathbb{R}$, the natural map

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k)^{(<h)} \to \mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)^{(<h)}$$

is an isomorphism. This last fact is key, because it essentially says that if we are working with U_p-eigensymbols of finite slope, then it doesn't matter if we use $\mathcal{D}_k$ or $\mathbf{D}_K$.

2.8. **Connecting to p-adic L-functions.** We close this section by mentioning a connection to p-adic L-functions (and thus to Stark-Heegner points).

Theorem 2.2. *Let f be a cuspidal eigenform on Γ_0 of non-critical slope,*[11] *and let φ_f be the corresponding modular symbol. If Φ_f is the unique overconvergent eigensymbol lifting φ_f (by Theorem 2.1), then $\Phi_f(\{\infty\} - \{0\})$ is the p-adic L-function of f.*

This theorem gives a construction of the p-adic L-function in one fell swoop as opposed to what was done in section 1.8 when the p-adic L-function was defined by gathering together the data of φ_f evaluated on infinitely many different divisors.

3. The Control Theorem: Comparing overconvergent modular symbols to classical ones

We aim to give an explicit proof the control theorem, at least for the slope 0 subspace.

3.1. **Finite approximation modules.** We would like to be able to approximate distributions in a systematic way (with a finite amount of data). Doing so would (a) allow us to represent distributions on a computer, and (b) lead us to an explicit proof of Steven's control theorem.

A first guess on how to form an approximation of a distribution μ with integral moments is to fix two integers M and N, and consider the first M moments of μ modulo p^N. Unfortunately, these approximations are not stable under the action of $\Sigma_0(p)$; that is, given such an approximation of μ, one cannot compute the corresponding approximation of $\mu\big|\gamma$ to the same accuracy. Indeed, the collection

[11] Note that we are assuming that f is an eigenform on Γ_0 and not on Γ. If one is starting off with a form on Γ, to form its p-adic L-function, one must choose a p-stabilization of this form to Γ_0. See section 4.1 for details.

of distributions whose first M moments vanish is not stable under the action of $\Sigma_0(p)$-stable. To see this, let's work out a little example.

Let $k = 0$ and let μ_4 denote the distribution which takes the value 1 on z^4 and 0 on all other monomials. Let $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ -p & 1 \end{smallmatrix}\right)$, and we compute

$$(\mu_4|\gamma)(z) = \mu_4\left(\frac{z}{1-pz}\right) = \mu_4\left(z \cdot \sum_{j=0}^{\infty}(pz)^j\right) = p^3$$

$$(\mu_4|\gamma)(z^2) = \mu_4\left(\frac{z^2}{(1-pz)^2}\right) = \mu_4\left(z^2 \cdot \left(\sum_{j=0}^{\infty}(pz)^j\right)^2\right) = 3p^2$$

$$(\mu_4|\gamma)(z^3) = \mu_4\left(\frac{z^3}{1-pz}\right) = \mu_4\left(z^3 \cdot \left(\sum_{j=0}^{\infty}(pz)^j\right)^3\right) = 3p$$

So even though the first 4 moments of μ_4 vanish, the same is not true of $\mu_4|\gamma$. However, do note that the early moments of $\mu_4|\gamma$ are highly divisible by p, and this divisibility trails off as we consider later moments.

This phenomenon holds quite generally. Let

$$\mathbf{D}_k^0 = \left\{\mu \in \mathbf{D}_k \mid \mu(x^j) \in \mathbb{Z}_p \text{ for all } j \geq 0\right\}$$

be the collection of distributions with all integral moments, and consider the subspace

$$\mathrm{Fil}^M \mathbf{D}_k^0 = \left\{\mu \in \mathbf{D}_k^0 \text{ such that } \mu(z^j) \in p^{M-j}\mathbb{Z}_p\right\}.$$

whose moments satisfy this trail-off of divisibility. That is, for $\mu \in \mathrm{Fil}^M \mathbf{D}_k^0$, the 0-th moment of μ is divisible by p^M, the first is divisible by p^{M-1}, the second is divisible by p^{M-2}, and so on. A direct computation (try it!) shows that $\mathrm{Fil}^M \mathbf{D}_k^0$ is stable under the weight k action of $\Sigma_0(p)$ for $k \geq 0$.

3.2. **Approximating distributions.** We now use the filtration $\left\{\mathrm{Fil}^M \mathbf{D}_k^0\right\}$ to systematically approximate distributions.

Definition 3.1. We define the M-th *finite approximation module* of $\mathbf{D}_k^0$ to be

$$\mathcal{F}_k(M) := \mathbf{D}_k^0 / \mathrm{Fil}^M(\mathbf{D}_k^0).$$

Proposition 3.2. *We have that $\mathcal{F}_k(M)$ is a $\Sigma_0(p)$-module and*

$$\begin{aligned} \mathcal{F}_k(M) &\xrightarrow{\sim} (\mathbb{Z}/p^M\mathbb{Z}) \times (\mathbb{Z}/p^{M-1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p\mathbb{Z}) \\ \overline{\mu} &\mapsto \left(\mu(z^j) + p^{M-j}\mathbb{Z}_p\right)_j \end{aligned}$$

is an isomorphism. In particular, $\mathcal{F}_k(M)$ is a finite set.

Proof. Since $\mathrm{Fil}^M(\mathbf{D}_k^0)$ is a $\Sigma_0(p)$-module, $\mathcal{F}_k(M)$ is also a $\Sigma_0(p)$-module. The fact that this map is an isomorphism follows directly from the definition of the filtration and the fact that the moment map identifies $\mathbf{D}_k^0$ with the set of sequences in $\mathbb{Z}_p$. □

By the above proposition, we can approximate $\mu \in \mathbf{D}_k^0$ with a finite amount of data by taking its image in $\mathcal{F}_k(M)$. Moreover, if one knows the image of μ in every $\mathcal{F}_k(M)$, then one can recover μ as one can recover all of its moments. Great!

3.3. **Lifting modular symbols.** Let's assume for simplicity that we are working in weight 2 (i.e. $k = 0$), and drop k from the notation. We seek to show that the specialization map

$$\mathrm{Symb}_{\Gamma_0}(\mathbf{D})^{\mathrm{ord}} \longrightarrow \mathrm{Symb}_{\Gamma_0}(\mathbb{Q}_p)^{\mathrm{ord}}$$

is an isomorphism. Here the superscript ord denotes the subspace where U_p acts invertibly (i.e. with slope 0), and so this statement is a special case of the control theorem.

To gain some intuition, let's assume this theorem is true, and from that, try to construct the unique lift of a given eigensymbol. Let φ denote some Hecke-eigensymbol in the target, $\mathrm{Symb}_{\Gamma_0}(\mathbb{Q}_p)^{\mathrm{ord}}$. Then take some overconvergent lift Ψ of φ in $\mathrm{Symb}_{\Gamma_0}(\mathbf{D})$. Note that we are **not** assuming that Ψ is a U_p-eigensymbol (and thus not necessarily in the slope zero subspace). Since the source of specialization is infinite-dimensional and the target is finite-dimensional, there will be lots and lots of choice for such a Ψ.

The operator U_p is compact, and so has an infinite collection of eigenvalues which tend to 0 in $\mathbb{Z}_p$. Say the eigenvalues are

$$\lambda_1, \lambda_2, \ldots, \lambda_n \ldots,$$

ordered by valuation, with corresponding eigensymbols,

$$\Phi_1, \Phi_2, \ldots \Phi_n \ldots,$$

with Φ_1 our sought after eigensymbol lifting φ. In particular, λ_1 is then the U_p-eigenvalue of φ.

Now write Ψ as an infinite linear combination of these symbols:

$$\Psi = \Phi_1 + a_2\Phi_2 + \cdots + \Phi_n + \ldots .$$ [12]

Applying the operator U_p/λ_1 repeatedly to Ψ, and looking at its eigen-expansion we get

$$(U_p/\lambda_1)^M\Psi = \Phi_1 + a_2(\lambda_2/\lambda_1)^M\Phi_2 + \cdots + (\lambda_n/\lambda_1)^M\Phi_n + \ldots .$$

Note that $(U_p/\lambda_1)^M\Psi$ is still a lift of φ as specialization is Hecke-equivariant – that is,

$$\rho^*((U_p/\lambda_1)^M\Psi) = (U_p/\lambda_1)^M\rho^*(\Psi) = (U_p/\lambda_1)^M\varphi = \varphi.$$

Since φ is an ordinary eigensymbol, λ_1 is a p-adic unit. To simplify matters, let's assume that λ_1 is the only unit eigenvalue of U_p.[13] In particular, as we continually apply U_p/λ_1, the higher terms in the eigen-expansion of Ψ get p-adically small, and thus we get a convergence:

$$\left\{(U_p/\lambda_1)^M\Psi\right\} \to \Phi_1,$$

and we have "constructed" the desired symbol.

To turn the above argument into a real proof, we need to (a) not assume the theorem we want to prove, and (b) deal with all of the convergence issues that arise in these eigen-expansions. We do this in the following steps.

[12] Because of convergence issues, there is no reason that such an expansion should even exist, but let's just imagine so anyway.

[13] If this were not true, one could use the other Hecke operators to kill off the other overconvergent eigensymbols of slope 0 in the expansion of Ψ without changing the fact that Ψ lifts φ.

A) **Prove that the specialization map is surjective.** We need this statement to know the existence of the lift Ψ of φ. This fact is not too hard to establish – the source is infinite-dimensional and the target is finite-dimensional, so the surjectivity statement is pretty reasonable.

B) **Prove directly that the sequence $\left\{\Psi|(U_p/\lambda)^M\right\}$ converges**. Here λ is the U_p-eigenvalue of φ. This also is not too hard. The underlying reason is the following lemma.

Lemma 3.3. *If Φ is in the kernel of specialization, then*

$$||\Phi|U_p|| \leq \frac{1}{p}||\Phi||.$$

In particular, any U_p-eigensymbol in the kernel of specialization has slope at least 1.

We'll assume this lemma for now (in fact it's just an easy computation). Back to the main argument, we'll check that $\left\{\Psi|(U_p/\lambda)^M\right\}$ is Cauchy. To this end, we note that $\Psi - \Psi|(U_p/\lambda)^j$ is in the kernel of specialization for any j (as both symbols lift φ). Thus,

$$\Psi|(U_p/\lambda_1)^{M_1} - \Psi|(U_p/\lambda_1)^{M_2} = \left(\Psi - \Psi|(U_p/\lambda_1)^{M_2-M_1}\right)\Big|(U_p/\lambda_1)^{M_1}$$

tends to 0 for M_1, M_2 large as λ_1 is a unit and the right hand side is U_p applied many times to an element in the kernel of specialization. This proves the desired Cauchy statement. Let Φ denote the limit of $\left\{\Psi|(U_p/\lambda)^M\right\}$.

C) **Prove that Φ is a Hecke-eigensymbol lifting φ.** That Φ lifts φ is clear as $(U_p/\lambda)^M\Psi$ lifts φ for every M. That Φ is an U_p-eigensymbol is clear as

$$\Phi|(U_p/\lambda) = \left(\lim_{M\to\infty}\Psi|(U_p/\lambda)^M\right)|U_p = \lim_{M\to\infty}\Psi|(U_p/\lambda)^{M+1} = \Phi.$$

The other eigenvalues are also easily checked. (Do it!)

3.4. **Lifting symbols – take II (a la M. Greenberg).** In the last section, we punted on the issue of simply lifting to φ to *some* overconvergent symbol – not even an eigensymbol. In fact, a lot was swept under the rug here (i.e. step A). It's not too hard to check this lift exists (using a little cohomology), but to directly write down a lift is involved (though completely worked out in [4]).

However, Matthew Greenberg found a method which sidesteps these difficulties and does steps A through C from the past section in one fell swoop (see [3]). Let's explain. Let φ be a U_p-eigensymbol in $\mathrm{Symb}_{\Gamma_0}(\mathbb{Q}_p)^{\mathrm{ord}}$. The idea is to successively lift φ to a U_p-eigensymbol in $\mathrm{Symb}_{\Gamma_0}(\mathcal{F}(M))^{\mathrm{ord}}$ for $M = 1, 2, \ldots$. Since $\varprojlim_M \mathcal{F}(M) = \mathbf{D}$ this would suffice to produce an eigenlifting of φ. Note also that this is exactly the kind of thing one would want to do if you were programming a computer.

Let's start with $M = 1$. To write down an element Ψ_1 of $\mathrm{Symb}_{\Gamma_0}(\mathcal{F}(1))^{\mathrm{ord}}$, we need to give the 0-th moment of $\Psi_1(D)$ modulo p^2 and the 1-st moment of $\Psi_1(D)$ modulo p for each divisor $D \in \Delta_0$. Since we are trying to write down a lift of φ, our hands our forced on the 0-th moments. Indeed, $\Psi_1(D)(\mathbf{1})$ should just be the reduction of $\varphi(D)$ modulo p^2. As for the 1-st moments, there is no clear choice. So just randomly assign values to $\Psi_1(D)(z) \in \mathbb{Z}/p\mathbb{Z}$.

The result is an element

$$\Psi_1 \in \mathrm{Maps}(\Delta_0, \mathcal{F}(1)).$$

Here Maps means simply that, set maps. We've lost the homomorphism property when we randomly assigned the first moments, and we've certainly lost the Γ-invariance.

To somehow fix this random choice, we apply U_p. (This shouldn't be so unreasonable considering the arguments from the last section.) Though, a word of warning here: U_p is a well-defined operator on modular symbols, i.e. it is independent of double coset representatives of $\Gamma_0 \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma_0$. On $\mathrm{Maps}(\Delta_0, \cdot)$, this is no longer true. So we just pick coset representations. That is U_p is *defined* as the operator

$$U_p := \sum_{a=0}^{p-1} \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right).$$

Now the magic: the element $\Phi_1 := \Psi_1 | (U_p/\lambda)$ lies in

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{F}(1)) \subseteq \mathrm{Maps}(\Delta_0, \mathcal{F}(1));$$

that is, Φ_1 is both additive and Γ_0-invariant. Moveover, Φ_1 is independent of any choices made!

Let's see why this is true. First, we'll check that Φ_1 is in fact a homomorphism. To see this, consider

$$\Phi_1(D) + \Phi_1(D') - \Phi_1(D + D') \tag{6}$$

$$= \left(\sum_{a=1}^{p-1} \Psi_1(\gamma_a D) + \Psi_1(\gamma_a D') - \Psi_1(\gamma_a D + \gamma_a D') \right) | \gamma_a \tag{7}$$

where $\gamma_a = \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)$. By construction, $\Psi_1(\gamma_a D) + \Psi_1(\gamma_a D') - \Psi_1(\gamma_a D + \gamma_a D')$ has vanishing 0-th moments as these moments are built out of the values of φ which is a bona-fide modular symbol. However, we have no control over the 1-st moments.

To understand what is going on, let's take an arbitrary element μ of $\mathcal{F}(1)$ with vanishing 0-th moment and act γ_a on it. We have

$$(\mu|\gamma_a)(\mathbf{1}) = \mu(\gamma_a \cdot \mathbf{1}) = \mu(\mathbf{1}) = 0,$$

and thus $\mu|\gamma_a$ still has vanishing first moment. Further,

$$(\mu|\gamma_a)(x) = \mu(\gamma_a \cdot x) = \mu(a + px) = \mu(a) + p\mu(x) = 0$$

as $\mu(a) = a\mu(\mathbf{1}) = 0$ and p kills whatever value $\mu(x)$ is taking. (Remember the 1-st moment lies in $\mathbb{Z}/p\mathbb{Z}$!) This means that any element in $\mathcal{F}(1)$ with vanishing first moment is killed by γ_a and hence the expression in (6) vanishes. In particular, Φ_1 is a homomorphism!

A similar argument proves that Φ_1 is Γ-invariant. Just consider $\Phi|\gamma - \Phi$ and argue just as before (remembering that $\varphi|\gamma = \varphi$).

And so we've done it! We've formed a lift of φ with values in $\mathcal{F}(1)$. To see how general this is, let's try to form a $\mathcal{F}(2)$-valued lift Ψ_2 of Φ_1. For any divisor $D \in \Delta_0$, we seek to define the 0-th, 1st, and 2nd moments of $\Psi_2(D)$ modulo p^3, p^2 and p respectively. The 0-th moment is easy – just reduce $\varphi(D)$ modulo p^3. For the 1st moment, we want to be lifting Φ_1. This means the value we choose for $\Psi_2(D)(x)$ should be congruent to $\Phi_1(D)(x)$ modulo p – pick any value modulo p^2 that works. Lastly, we have no info on 2nd moment, so again, pick randomly.

As before, we set $\Phi_2 := \Psi_2|U_p \in \mathrm{Maps}(\Delta_0, \mathcal{F}(2))$. To check that Φ_2 is a homomorphism, the identical argument as above reduces us to checking that any element of $\mathcal{F}(2)$ with 0-th moment equal to 0 and first moment divisible by p is killed by γ_a. Let's compute!

Fix μ any such element of $\mathcal{F}(2)$. We have

$$(\mu|\gamma_a)(\mathbf{1}) = \mu(\gamma_a \cdot \mathbf{1}) = \mu(\mathbf{1}) = 0,$$

Further,

$$(\mu|\gamma_a)(x) = \mu(\gamma_a \cdot x) = \mu(a + px) = \mu(a) + p\mu(x) = p\mu(x) = 0$$

as $\mu(x)$ is divisible by p and thus $p\mu(x)$ is 0 modulo p^2. Lastly,

$$(\mu|\gamma_a)(x^2) = \mu(\gamma_a \cdot x) = \mu((a + px)^2) = \mu(a^2) + 2ap\mu(x) + p^2\mu(x^2) = 0$$

as before as desired.

As you might imagine, this just keeps working. The underlying fact that is needed is that if μ is in $\mathcal{F}(M)$ with vanishing 0-th moment and with vanishing projection to $\mathcal{F}(M-1)$ then $\mu|\gamma_a = 0$. We leave the details to you (plus the generalizations to higher weight and higher (non-critical) slope).

4. Overconvergent modular symbols and p-adic L-functions

4.1. **Motivating the construction of p-adic L-functions.** The p-adic L-function of a eigenform $f \in S_2(\Gamma_0(N))$ is a distribution $\mu_f \in \mathcal{D}$ such that when μ_f is evaluated at some Dirichlet character χ the result should be $\dfrac{L(f,\chi,1)}{\Omega_f^{\pm}}$ up to some explicit controllable constants.

When $p \nmid N$ and f is a p-ordinary form (that is, when $a_p(f)$ is a p-adic unit), a formula was given in section 1.8 for $\mu_f(a + p^n\mathbb{Z}_p)$.[14] We now take a little bit of time to motivate this formula, and in the process, derive the basic properties of μ_f.

The starting point is equation (3) which relates L-values to period integrals. Rewriting this formula is terms of modular symbols gives

$$\tau(\overline{\chi})\frac{L(f,\chi,1)}{\Omega_f^{\pm}} = \sum_{a \bmod p^n} \overline{\chi}(a) \cdot \varphi_f^{\pm}(\{\infty\} - \{a/p^n\}). \tag{8}$$

Staring at this above expression just right, one sees the right hand side as a Riemann sum. Indeed, think of $\overline{\chi}$ as a function on $\mathbb{Z}_p$, and cover $\mathbb{Z}_p$ by the opens $a + p^n\mathbb{Z}_p$. Then think of $\varphi_f^{\pm}(\{\infty\} - \{a/p^n\})$ as the measure of $a + p^n\mathbb{Z}_p$. The right hand side then reads as the sum over opens in our cover of the measure of that open times the value of the function on that open. Moreover, this Riemann sum equals an L-value, which is exactly what we are after.

However, if μ is a measure on $\mathbb{Z}_p$, then it must be true that

$$\mu(a + p^n\mathbb{Z}_p) = \sum_{j=0}^{p-1} \mu(a + jp^n + p^{n+1}\mathbb{Z}_p) \tag{9}$$

as $a + p^n\mathbb{Z}_p$ is the disjoint union of opens of the form $a + jp^n + p^{n+1}\mathbb{Z}_p$. Could it be that the values of the modular symbol $\varphi_f^{\pm}(\{\infty\} - \{a/p^n\})$ satisfy such an additivity

[14]Writing $\mu_f(a + p^n\mathbb{Z}_p)$ is a bit of an abusive of notation. What is meant here is the value obtained when evaluating μ_f on the characteristic function of $a + p^n\mathbb{Z}_p$.

property? Amazingly the answer is nearly yes, and results from the fact that φ_f is a T_p-eigensymbol.

Indeed, we know that $\varphi_f | T_p = a_p \varphi_f$. Thus,

$$\begin{aligned}
&a_p \cdot \varphi_f(\{\infty\} - \{a/p^n\}) \\
&\quad = (\varphi_f | T_p)(\{\infty\} - \{a/p^n\}) \\
&\quad = (\varphi_f | \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right))(\{\infty\} - \{a/p^n\}) + \sum_{j=0}^{p-1} (\varphi_f | \left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right))(\{\infty\} - \{a/p^n\}) \\
&\quad = \varphi_f(\{\infty\} - \{a/p^{n-1}\}) + \sum_{a=0}^{p-1} \varphi_f(\{\infty\} - \{(a + jp^n)/p^{n+1}\}).
\end{aligned}$$

Note that this formula is nearly what we are looking for except for two points: (a) the factor of a_p in the front of the left hand side, and (b) the first term of the right hand side which arose from acting by $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Let's start with the second issue. We first note that if we had been using the Hecke operator U_p instead of T_p, then this problem wouldn't be present. Indeed, U_p is defined by only p terms and is exactly missing that extra troublesome matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The reason we are using T_p instead of U_p is that we are working at a level prime to p. To get around this, we can force p into the level through the process of p-stabilization.

Namely, our form f is an eigenform for the full Hecke algebra for level $\Gamma_0(N)$. However, if we instead think of f as an eigenform with level $\Gamma_0(Np)$, it would no longer be an eigenform at p (although it is still an eigenform away from p). As usual, we also consider the form $f(pz)$ which has level $\Gamma_0(Np)$ and is an eigenform away from p. Moreover, the span of $f(z)$ and $f(pz)$ is stable under the action of U_p, and a pleasant computation shows that the characteristic polynomial of U_p on this two-dimensional space is nothing other than the Hecke polynomial $x^2 - a_p x + p$.

Performing a little linear algebra (i.e. diagonalizing), we see that if α and β are the roots of $x^2 - a_p x + p$, and if we set

$$f_\alpha = f(z) - \beta f(pz) \text{ and } f_\beta = f(z) - \alpha f(pz),$$

then $f_\alpha | U_p = \alpha f_\alpha$ and $f_\beta | U_p = \beta f_\beta$ – i.e. f_α and f_β are U_p-eigenforms with eigenvalues α and β respectively. Since we are assuming that f is p-ordinary, a_p is a p-adic unit. Thus exactly one of the two roots α and β is also a p-adic unit as their sum is a_p and their product is p. Let α be this unit root.

Returning to modular symbols, we can consider the modular symbol φ_{f_α} attached to f_α. Since f_α is a U_p-eigensymbol, we get the following formula (analogous to what we derived above but without the extra bothersome term):

$$\alpha \cdot \varphi_{f_\alpha}(\{\infty\} - \{a/p^n\}) = \sum_{a=0}^{p-1} \varphi_{f_\alpha}(\{\infty\} - \{(a + jp^n)/p^{n+1}\}).$$

This relation is nearly what we had hoped for except for the the presence of the α on the left hand side. But, this is easily dealt with. Indeed, we set

$$\mu_f(a + p^n \mathbb{Z}_p) = \frac{1}{\alpha^n} \varphi_{f_\alpha}(\{\infty\} - \{a/p^n\}),$$

and an easy computation shows that μ_f satisfies the additivity relation in (9).[15] Further, a simple computation shows that $\varphi_{f_\alpha} = \varphi_f - \frac{1}{\alpha}\varphi_f | \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Plugging this into the definition of μ_f yields the formula stated in section 1.8. Lastly, the interpolation property follows directly from (8).

4.2. **Proving that $\Phi(\{0\} - \{\infty\})$ is the p-adic L-function.** In this section, we prove Theorem 2.2 for weight 2 ordinary forms. Note that in the statement of that theorem, f has level $\Gamma_0(Np)$. To match the notation of the previous seciton, we will take f to be f_α. That is, let Φ be the unique overconvergent modular symbol in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D})$ lifting φ_{f_α}. We will show that $\Phi(\{0\} - \{\infty\}) = \mu_f$.

Note that

$$\begin{aligned}\Phi(\{\infty\} - \{0\}) &= \alpha^{-n}(\Phi | U_p^n)(\{\infty\} - \{0\}) \\ &= \alpha^{-n} \sum_{a=0}^{p^n-1} \Phi\left(\{\infty\} - \{a/p^n\}\right) \Big| \left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right).\end{aligned}$$

Evaluating at the characteristic function of $a + p^n\mathbb{Z}_p$ gives

$$\begin{aligned}&\Phi(\{\infty\} - \{0\})(\mathbf{1}_{a+p^n\mathbb{Z}_p}) \\ &\qquad = \alpha^{-n}\left(\Phi\left(\{\infty\} - \{a/p^n\}\right) \Big| \left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right)\right)(\mathbf{1}_{a+p^n\mathbb{Z}_p}) \\ &\qquad = \alpha^{-n}\Phi\left(\{\infty\} - \{a/p^n\}\right)\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) \cdot \mathbf{1}_{a+p^n\mathbb{Z}_p}\right) \\ &\qquad = \alpha^{-n}\Phi\left(\{\infty\} - \{a/p^n\}\right)\left(\mathbf{1}_{\mathbb{Z}_p}\right).\end{aligned}$$

But the specialization map for $k = 0$ is simply taking total measure. Thus, we get

$$\Phi(\{\infty\} - \{0\})(\mathbf{1}_{a+p^n\mathbb{Z}_p}) = \alpha^{-n}\rho^*(\Phi)\left(\{\infty\} - \{a/p^n\}\right) = \alpha^{-n}\varphi_{f_\alpha}\left(\{\infty\} - \{a/p^n\}\right)$$

which agrees exactly with the definition of the p-adic L-function from the previous section.

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[15] In fact, this definition only tells us the value of μ_f on locally constant functions. However, since the values given are bounded (α is a unit!), standard arguments via Riemann sums allow us to integrate μ_f against any continuous function (and thus against any locally analytic function).