INTRODUCTION TO WILD RAMIFICATION OF SCHEMES AND SHEAVES

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1. Brief summary on étale cohomology

In this section, k denotes a field, a scheme will mean a separated scheme of finite type over k and a morphism of schemes will mean a morphism over k. We put $p = \text{char } k$.

1.1. **Definition and examples of étale sheaves.** A morphism $X \to Y$ of schemes is said to be étale if $\Omega^1_{X/Y} = 0$ and if X is flat over Y.

Example 1.1. *An open immersion is étale.*

The morphism $\mathbf{G}_m = \text{Spec } k[T^{\pm 1}] \to \mathbf{G}_m$ *defined by* $T \mapsto T^m$ *is étale if and only if* m *is invertible in* k.

The morphism $\mathbf{G}_a = \text{Spec } k[T] \to \mathbf{G}_a$ *defined by* $T \mapsto T^p - T$ *is étale if* $p > 0$.

A family $(U_i \to X)_{i \in I}$ of étale morphisms is called an étale covering if $X = \bigcup_{i \in I} \text{Image}(U_i \to X)$.

A contravariant functor (= presheaf) $\mathcal{F}$: (Etale schemes over X) $\to$ (Sets) (or $\to$ (Abelian groups)) is called a sheaf if it satisfies the patching conditions

$$\mathcal{F}(U) \xrightarrow{\simeq} \text{Ker}\left(\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_{ij})\right)$$

for every étale covering $(U_i \to U)_{i \in I}$. Here $U_{ij} = U_i \times_U U_j$ and Ker denotes the set (or the abelian group)

$$\left\{ (s_i) \in \prod_{i \in I} \mathcal{F}(U_i) \;\middle|\; s_i|_{U_{ij}} = s_j|_{U_{ij}} \text{ for every } i, j \in I \right\}.$$

Example 1.2. *A representable functor* $\text{Hom}(-, Y)$ *is a sheaf. In particular, a constant sheaf, the additive sheaf* $\mathbf{G}_a$*, the mulitiplicative sheaf* $\mathbf{G}_m$ *etc. are actually sheaves.*

A morphism $f : X \to Y$ of schemes defines the push-forward functor (Etale sheaves/X) $\to$ (Etale sheaves/Y) by $f_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$. It has a left adjoint functor f^*. For an open immersion $j\colon U \to X$, the extension by 0 is defined by $j_!\mathcal{F} = \mathrm{Ker}(j_*\mathcal{F} \to i_*i^*j_*\mathcal{F})$.

We will use the same notation for a scheme and for the corresponding representable functor. An étale sheaf representable by a finite étale scheme is called a locally constant constructible sheaf.

For a group scheme G over X, we say an étale sheaf T with G-action $G \times T \to T$ is an G-torsor if étale locally on X, T is isomorphic to G with the canonical action of G.

Example 1.3. *For an invertible $\mathcal{O}_X$-module $\mathcal{L}$, the sheaf $\mathcal{I}som(\mathcal{O}_X, \mathcal{L})$ is a $\mathbf{G}_m$-torsor. Conversely, a $\mathbf{G}_m$-torsor on X defines an invertible $\mathcal{O}_X$-module by flat descent.*

1.2. **Etale cohomology.** Etale sheaves of abelian groups on a scheme X form an abelian category with enough injectives and hence the right derived functor $H^i(X, -)$ of the left exact functor $\Gamma(X, -)$ is defined. The compact support cohomology $H^i_c(X, -)$ is defined as $H^i(\bar{X}, j_!-)$ by taking a compactification $j\colon X \to \bar{X}$. If X itself is proper, we have $H^i_c(X, -) = H^i(X, -)$.

If G is a commutative group scheme on X, the set of isomorphism classes of G-torsors on X is canonically idenitified with $H^1(X, G)$. In particular, $H^1(X, \mathbf{G}_m)$ is canonically identified with the Picard group $\mathrm{Pic}(X)$ defined as the group of isomorphism classes of invertible $\mathcal{O}_X$-modules. For an integer n invertible in k, the Kummer sequence $0 \to \mu_n \to \mathbf{G}_m \overset{t\mapsto t^n}{\to} \mathbf{G}_m \to 0$ induces an exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X)^\times/(\Gamma(X, \mathcal{O}_X)^\times)^n \to H^1(X, \mu_n) \to \mathrm{Pic}(X)[n] \to 0. \tag{1}$$

Example 1.4. *If X is a proper smooth geometrically connected curve, it gives an isomorphism*

$$H^1(X_{\bar{k}}, \mu_n) \to \mathrm{Jac}_X(\bar{k})[n] \tag{2}$$

where Jac_X denotes the Jacobian variety of X. Further, we have an isomorphism

$$\mathrm{Pic}(X_{\bar{k}})/n\mathrm{Pic}(X_{\bar{k}}) = \mathbf{Z}/n\mathbf{Z} \to H^2(X_{\bar{k}}, \mu_n) \tag{3}$$

and vanishing $H^q(X_{\bar{k}}, \mu_n) = 0$ for $q > 2$.

1.3. **Fundamental group.** For a geometric point $\bar{x}$ of a connected scheme X, the fundamental group $\pi_1(X, \bar{x})$ is defined by requiring that the fiber functor

$$\begin{aligned} &(\text{Finite étale schemes}/X) \\ &\quad \to (\text{Finite sets with continuous action of } \pi_1(X, \bar{x})) \end{aligned} \tag{4}$$

defined by $X \to X(\bar{x})$ is an equivalence of categories. If we identify a commutative finite étale scheme A over X with a finite abelian group A with the action of $\pi_1(X, \bar{x})$, the étale cohomology $H^1(X, A)$ is identified with the cohomology $H^1(\pi_1(X, \bar{x}), A)$ of the profinite group $\pi_1(X, \bar{x})$.

We fix a prime number ℓ different from the characteristic of k. We call an inverse system $\mathcal{F} = (\mathcal{F}_n)$ of locally constant constructible sheaves of free $\mathbf{Z}/\ell^n\mathbf{Z}$-modules satisfying $\mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \xrightarrow{\simeq} \mathcal{F}_n$ for every n a smooth $\mathbf{Z}_\ell$-sheaf. If X is connected, the equivalence (4) of categories induces an equivalence of categories

$$(\text{Smooth } \mathbf{Z}_\ell\text{-sheaves}/X) \to (\text{Cont. } \mathbf{Z}_\ell\text{-representations of } \pi_1(X, \bar{x})).$$

1.4. **The Euler number.** For a smooth $\mathbf{Q}_\ell$-sheaf $\mathcal{F} \otimes \mathbf{Q}_\ell$, its cohomology and the compact support cohomology are defined by $H^i(X_{\bar{k}}, \mathcal{F} \otimes \mathbf{Q}_\ell) = \varprojlim_n H^i(X_{\bar{k}}, \mathcal{F}_n) \otimes \mathbf{Q}_\ell$ and $H^i_c(X_{\bar{k}}, \mathcal{F} \otimes \mathbf{Q}_\ell) = \varprojlim_n H^i_c(X_{\bar{k}}, \mathcal{F}_n) \otimes \mathbf{Q}_\ell$. They are known to be a $\mathbf{Q}_\ell$-vector space of finite dimension and known to be 0 for $i > 2\dim X$. The Euler numbers are defined by

$$\chi(X_{\bar{k}}, \mathcal{F}) = \sum_{i=0}^{2\dim X} (-1)^i \dim H^i(X_{\bar{k}}, \mathcal{F}),$$

$$\chi_c(X_{\bar{k}}, \mathcal{F}) = \sum_{i=0}^{2\dim X} (-1)^i \dim H^i_c(X_{\bar{k}}, \mathcal{F})$$

respectively.

Example 1.5. *If X is a proper smooth curve of genus g, we have $\dim H^0(X_{\bar{k}}, \mathbf{Q}_\ell) = 1, \dim H^1(X_{\bar{k}}, \mathbf{Q}_\ell) = 2g, \dim H^2(X_{\bar{k}}, \mathbf{Q}_\ell) = 1$ and $\chi(X_{\bar{k}}, \mathbf{Q}_\ell) = 2 - 2g$. If $U = X \setminus D$ is the complement of a finite étale divisor D of degree $d > 0$ in a proper smooth curve X of genus g, we have $\chi_c(U_{\bar{k}}, \mathbf{Q}_\ell) = 2-2g-d$ by the long exact sequence $\to H^i_c(U_{\bar{k}}, \mathbf{Q}_\ell) \to H^i(X_{\bar{k}}, \mathbf{Q}_\ell) \to H^i(D_{\bar{k}}, \mathbf{Q}_\ell) \to$.*

If X is a proper smooth scheme of dimension d, we have an equality $\chi(X_{\bar{k}}, \mathbf{Q}_\ell) = (-1)^d \deg c_d(\Omega^1_{X/k})$ with the degree of the top Chern class. If $U = X \setminus D$ is the complement of a divisor D with normal crossings in a proper smooth scheme X of dimension d, we have $\chi_c(U_{\bar{k}}, \mathbf{Q}_\ell) = \deg(-1)^d c_d(\Omega^1_{X/k}(\log D))$.

2. Formula for the Euler number

In this section, we keep the assumption that k is a field and schemes are separated of finite type over k. We discuss the following problem.

Problem. *How do we compute $\chi_c(U_{\bar{k}}, \mathcal{F})$ for a smooth $\mathbf{Q}_\ell$-sheaf on a smooth scheme U?*

In fact, we will be interested in computing the difference $\chi_c(U_{\bar{k}}, \mathcal{F}) - \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbf{Q}_\ell)$. If char $k = 0$, we have $\chi_c(U_{\bar{k}}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbf{Q}_\ell)$. In the following, we assume k is a perfect field of characteristic $p > 0$.

2.1. **The Grothendieck-Ogg-Shafarevich formula.** If $\dim U = 1$, we know a classical formula.

Theorem 2.1 (Grothendieck-Ogg-Shafarevich formula [11])**.** *Let X be a smooth proper curve and U be a dense open subscheme. Then, for a smooth $\mathbf{Q}_\ell$-sheaf on U, we have*

$$\chi_c(U_{\bar{k}}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbf{Q}_\ell) - \sum_{x \in X \setminus U} \text{Sw}_x \mathcal{F}. \tag{5}$$

Example 2.1. *Let $\pi \colon V \to U$ be a finite etale morphism and $\mathcal{F} = \pi_* \mathbf{Q}_\ell$ be the locally constant sheaf corresponding to the induced representation $\text{Ind}_{\pi_1(V)}^{\pi_1(U)} \mathbf{Q}_\ell$. Let X and Y denote the smooth compactifications of U and V respectively and $\bar{\pi} \colon Y \to X$ be the induced morphism. let $D \subset X$ and $E \subset Y$ denote the reduced divisors such that the complements are U and V respectively. Then,* (5) *for $\mathcal{F} = \pi_* \mathbf{Q}_\ell$ gives*

$$\begin{aligned}\chi_c(V, \mathbf{Q}_\ell) =& [V : U] \cdot \chi_c(U, \mathbf{Q}_\ell) \\ & - \sum_{x \in X \setminus U} \sum_{y \in \pi^{-1}(x)} \text{length}_{\mathcal{O}_{Y,y}} (\Omega^1_Y(\log E) / \bar{\pi}^* \Omega^1_X(\log D))_y.\end{aligned}$$

This gives a sheaf-theoretic reformulation of the Riemann-Hurwitz formula

$$2g_Y - 2 = [Y : X](2g_X - 2) + \sum_{y \in Y \setminus V} \text{length}_{\mathcal{O}_{Y,y}} \Omega^1_{Y/X,y}.$$

2.2. **Swan conductor.** We recall the definition of the Swan conductor $\text{Sw}_x \mathcal{F}$ from two points of view, corresponding to the upper and the lower ramification groups respectively (see Section 3).

Let K denote the local field at x. Namely the fraction field of the completion of the local ring $\mathcal{O}_{X,x}$. Let $\bar{K}$ be a separable closure of K and $G_K = \text{Gal}(\bar{K}/K)$ be the absolute Galois group. Then, by the map Spec $K \to U$, the pull-back of the ℓ-adic representation of $\pi_1(U, \text{Spec } \bar{K})$ corresponding to $\mathcal{F}$ defines an ℓ-adic representation V of G_K.

The group G_K has the inertia subgroup $I_K = \text{Gal}(\bar{K}/K^{\text{ur}})$ and its pro-p Sylow subgroup $P_K = \text{Gal}(\bar{K}/K^{\text{tr}})$ corresponding to the maximal unramified extension K^{ur} and the maximal unramified extension $K^{\text{tr}} = K^{\text{ur}}(\pi^{1/m}; p \nmid m)$ respectively.

The wild inertia group P_K has a decreasing filtration by ramification groups (G_K^r) indexed by positive rational numbers $r > 0$. Since the action of the pro-p-group P_K on V factors through a finite quotient, the filtration defines a decomposition $V = \bigoplus_{r\geq 0} V^{(r)}$ characterized by $(V^{(r)})^{G_K^r} = 0$ for $r > 0$, $(V^{(r)})^{G_K^s} = V^{(r)}$ for $0 < r < s$ and $V^{P_K} = V^{(0)}$. The Swan conductor $\mathrm{Sw}_x\mathcal{F} = \mathrm{Sw}_K V$ is defined to be $\sum_{r>0} r \cdot \dim V^{(r)}$. The equality $\mathrm{Sw}_K V = 0$ is equivalent to $V = V^{(0)}$ that means the action of P_K on V is trivial.

To explain another description of the definition, we make an extra assumption that the action of G_K on V factors through a finite quotient G corresponding to a finite Galois extension L over K. For $\sigma \neq 1, \in G$, we put

$$s_{L/K}(\sigma) = -\text{length } \mathcal{O}_L \Big/ \left(\frac{\sigma(a)}{a} - 1; a \in \mathcal{O}_L, \neq 0\right) \tag{6}$$

The integer $s_{L/K}(\sigma)$ is 0 unless σ is not an element of the image $P \subset G$ of P_K. We define $s_{L/K}(1)$ by requiring $\sum_{\sigma\in G} s_{L/K}(\sigma) = 0$. Then, the Swan conductor $\mathrm{Sw}_K V$ is defined by

$$\mathrm{Sw}_K V = \frac{1}{|I|} \sum_{\sigma\in P} s_{L/K}(\sigma) \cdot \mathrm{Tr}(\sigma : V) \tag{7}$$

where $I \subset G$ denotes the image of P_K.

2.3. **Log product.** We formulate a generalization of Theorem 2.1 to higher dimension by giving a geometric interpretation of the Swan character $s_{L/K}(\sigma)$.

Let U be a smooth separated scheme of finite type of dimension d over k. For a separated scheme S of finite type over k, the Chow group $CH_0(S)$ denotes the group of 0-cycles modulo rational equivalence. We will give a definition of the *Swan class* $\mathrm{Sw}\ \mathcal{F}$ as an element of $CH_0(X \setminus U)_{\mathbb{Q}(\zeta_{p^\infty})} = CH_0(X \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}(\zeta_{p^\infty})$ for a compactification X of U under some extra simplifying assumptions.

For a finite etale Galois covering $V \to U$ of Galois group G, we define the Swan character class

$$s_{V/U}(\sigma) \in CH_0(Y \setminus V)$$

for $\sigma \in G$ assuming that Y is a smooth compactification of V satisfying certain good properties. We refer to [14, Definition 4.1] for the definition in the general case that requires alteration.

Assume Y is a proper smooth scheme containing V as the complement of a divisor D with simple normal crossings. Let $D_1, \ldots, D_n$ be the irreducible components of D and let $(Y \times_k Y)' \to Y \times_k Y$ be the blow-up at $D_i \times_k D_i$ for every $i = 1, \ldots, n$. Namely the blow-up by the product of the ideal sheaves $\mathcal{I}_{D_i\times_k D_i} \subset \mathcal{O}_{Y\times_k Y}$. We call the complement $Y *_k Y \subset (Y\times_k Y)'$ of the proper transform of $(D\times_k Y)\cup(Y\times_k D)$ the log product. The diagonal map $\delta\colon Y\to Y\times_k Y$ is uniquely lifted to a closed immersion $\tilde{\delta}\colon Y\to Y *_k Y$ called the log diagonal. We introduce the log product in order to focus on the wild ramification.

Example 2.2. Assume $X = \operatorname{Spec} k[T_1,\ldots,T_d]$ and D is defined by $T_1\cdots T_n$ for $0\leq n\leq d$. Then, the log product $P = X *_k X$ is the spectrum of

$$A = k[T_1,\ldots,T_d,S_1,\ldots,S_d,U_1^{\pm 1},\ldots,U_n^{\pm 1}]/(S_1-U_1T_1,\ldots,S_n-U_nT_n) \tag{8}$$

and the log diagonal $\tilde{\delta}\colon X\to P = X *_k X$ is defined by $U_1=\cdots=U_n=1$ and $T_{n+1}=S_{n+1},\ \ldots,\ T_d=S_d$.

2.4. **Swan character class and an open Lefschetz trace formula.** Let $\sigma\in G$ be an element different from the identity and let Γ be a closed subscheme of $Y *_k Y$ of dimension $d=\dim Y$ such that the intersection $\Gamma\cap(V\times_k V)$ is equal to the graph Γ_σ of σ. By the assumption that V is etale over U, the intersection $\Gamma_\sigma\cap\Delta_V$ with the diagonal $\Delta_V = \delta(V)\subset V\times_k V$ is empty. Hence the intersection product $(\Gamma,\Delta_Y^{\log})_{Y *_k Y}$ with the log diagonal $\Delta_Y^{\log}=\tilde{\delta}(Y)\subset Y *_k Y$ is defined in $CH_0(Y\setminus V)$.

The intersection product $(\Gamma,\Delta_Y^{\log})_{Y *_k Y}$ is shown to be independent of the choice of Γ under the assumption that $V\to U$ is extended to a map $Y\to X$ to a proper scheme X over k containing U as the complement of a Cartier divisor B and that the image of Γ in the log product $X *_k X$ defined with respect to B is contained in the log diagonal $\Delta_X^{\log}$.

The Swan chararacter class $s_{V/U}(\sigma)\in CH_0(Y\setminus V)$ for $\sigma\neq 1$ is defined by

$$s_{V/U}(\sigma) = -(\Gamma,\Delta_Y^{\log})_{Y *_k Y}. \tag{9}$$

For $\sigma=1$, it is defined by requiring $\sum_{\sigma\in G} s_{V/U}(\sigma)=0$. For $\sigma\neq 1$, we have

$$\sum_{q=0}^{2\dim V}(-1)^q\operatorname{Tr}(\sigma^*\colon H^q_c(V_{\bar{k}},\mathbb{Q}_\ell)) = -\deg_k s_{V/U}(\sigma) \tag{10}$$

by a Lefschetz trace formula for open varieties [14, Theorem 2.3.4] for a prime number ℓ different from the characteristic of k.

Example 2.3. Assume that V is a curve and let Y be a smooth compactification. We have $CH_0(Y\setminus V)=\bigoplus_{y\in Y\setminus V}\mathbb{Z}$. For $\sigma\neq 1,\in G$, we have

(11)

$$s_{V/U}(\sigma) = -\sum_{y\in\{y\in Y|\sigma(y)=y\}} \text{length } \mathcal{O}_y \Big/ \left(\frac{\sigma(a)}{a} - 1; a \in \mathcal{O}_y, \neq 0\right) \cdot [y].$$

2.5. **Swan class and the generalization of the GOS formula.** Let ℓ be a prime number different from $p = \operatorname{char} k > 0$. We consider a smooth ℓ-adic sheaf $\mathcal{F}$ on U and define the Swan class $\mathrm{Sw}_U\mathcal{F} \in CH_0(X \setminus U)_{\mathbb{Q}(\zeta_{p^\infty})}$. Here we only give a definition assuming that there exist a finite etale Galois covering $f\colon V \to U$ trivializing $\mathcal{F}$ and a good compactification Y of V as above.

We refer to [14, Definition 4.2.2] for the definition in the general case that requires reduction modulo ℓ and Brauer traces. Let G denote the Galois group $\mathrm{Gal}(V/U)$ and M be the representation of G corresponding to $\mathcal{F}$. Then, the Swan class is defined by

$$\mathrm{Sw}_U\mathcal{F} = \frac{1}{|G|}\sum_{\sigma\in G} f_* s_{V/U}(\sigma) \cdot \mathrm{Tr}(\sigma : M). \tag{12}$$

By the equality (11), this is an immediate generalization of the classical definition (7).

The Lefschetz trace formula for open varieties (10) implies the following generalization of the Grothendieck-Ogg-Shafarevich formula:

Theorem 2.2 ([14, Theorem 4.2.9])**.** *Let U be a separated smooth scheme of finite type over k. For a smooth ℓ-adic sheaf $\mathcal{F}$ on U, we have*

$$\chi_c(U_{\bar k}, \mathcal{F}) = \operatorname{rank} \mathcal{F} \cdot \chi_c(U_{\bar k}, \mathbb{Q}_\ell) - \deg_k \mathrm{Sw}_U\mathcal{F}. \tag{13}$$

3. Ramification groups of a local field

We discuss a geometric definition of the filtration by ramification groups of Galois groups of local fields. Let K be a complete discrete valuation field with not necessarily perfect residue field $F = \mathcal{O}_K/\mathfrak{m}_K$.

3.1. **The lower and the upper ramification groups.** For a finite Galois extension L over K, the Galois group $G = \mathrm{Gal}(L/K)$ has two decreasing filtrations, the lower numbering filtration $(G_i)_{i\in\mathbb{N}}$ and the upper numbering filtration $(G^r)_{r\in\mathbb{Q},>0}$.

In the classical case where the residue field is perfect, they are the same up to renumbering by the Herbrand function [1, Chapitre IV Section 3]. However, their properties make good contrasts. The lower one has an elementary definition and is compatible with subgroups while the upper one has more sophisticated definition and is compatible with quotients. The lower one is simply defined by $G_i = \mathrm{Ker}(G \to$

$\mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L^i))$. More geometrically, it is rephrased by using rigid geometry as follows.

3.2. **Rigid geometric picture.** Take a presentation

$$\mathcal{O}_K[X_1, \ldots, X_n]/(f_1, \ldots, f_n) \to \mathcal{O}_L$$

of the integer ring of L. We consider the n-dimensional closed disk D^n defined by $\|x\| \leq 1$ over K in the sense of rigid geometry and the morphism of disks $f\colon D^n \to D^n$ defined by $f_1, \ldots, f_n$. Then the Galois group G is identified with the inverse image $f^{-1}(0)$ of the origin $0 \in D^n$. In other words, we have a cartesian diagram

$$\begin{CD} G @>>> D^n \\ @VVV @VV{f}V \\ \{0\} @>>> D^n. \end{CD} \tag{14}$$

The subgroups G_i and G^r are defined to consist of the points of G that are *close* to the identity in certain senses. For the lower one, the closeness is simply measured by the distance. Namely, the lower numbering subgroup $G_i \subset G$ consists of the points $\sigma \in G$ satisfying $d(\sigma, \mathrm{id}) \leq \|\pi_L^i\|$ for a prime element π_L of L.

To define the upper numbering filtration, we consider, for a rational number $r > 0$, the inverse image $V_r = \{x \in D^n \mid d(f(x), 0) \leq \|\pi_K\|^r\} \subset D^n$ of the closed subdisk of radius $\|\pi_K\|^r$, as an affinoid subdomain containing G. The upper numbering subgroup G^r consists of the points in G contained in the same geometric connected component of V_r as the identity.

3.3. **Interpretation via schemes.** In the following, we give a definition of of a logarithmic variant of the upper numbering filtration only using schemes, under the following extra assumption:

(G) There exist a smooth scheme X over a perfect field k, a smooth irreducible divisor D of X and an isomorphism $\mathcal{O}_K \to \widehat{\mathcal{O}}_{X,\xi}$ to the completion of the local ring at the generic point ξ of D.

In the description using rigid geometry above, a key construction is shrinking of the radius. For schemes, the corresponding construction is the blow-up.

Let L be a finite Galois extension of K of Galois group G. Then, by replacing X by an stale neighborhood of ξ if necessary, there exist

G-torsor V over the complement $U = X \setminus D$ and a cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}\ L & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ K & \longrightarrow & U \end{array}$$

of G-torsors such that the normalization Y of X in $V \to U$ is smooth over k and finite flat over X and the reduced inverse image E of D is a smooth divisor of Y.

In the product $X \times_k S$, we have divisors $D \times_k S$ and $X \times_k D_S$ where $D_S = \mathrm{Spec}\ F \subset S = \mathrm{Spec}\ \mathcal{O}_K$ denotes the closed point. We consider the blow-up $(X \times_k S)'$ of $X \times_k S$ at their intersection $D \times_k D_S$ and define the log product $P = X *_k S \subset (X \times_k S)'$ to be the complement of the proper transforms of $D \times_k S$ and $X \times_k D_S$. The generic fiber $P \times_S \mathrm{Spec}\ K$ is $U \times_k \mathrm{Spec}\ K$. Let Q denote the normalization of P in the finite etale covering $V \times_k \mathrm{Spec}\ K$ of $U \times_k \mathrm{Spec}\ K$.

The canonical map $S \to X$ is uniquely lifted to a section $s\colon S \to P$. In the cartesian diagram

(15)
$$\begin{array}{ccc} T & \xrightarrow{i} & Q \\ \downarrow & & \downarrow f \\ S & \xrightarrow{s} & P \end{array}$$

we have $T = \mathrm{Spec}\ \mathcal{O}_L$ and the vertical arrows are finite flat. This diagram should be regarded as a scheme theoretic counterpart of (14).

We consider a finite separable extension K' of K containing L as a subextension, in order to make a base change. We put $S' = \mathrm{Spec}\ \mathcal{O}_{K'}$, $F' = \mathcal{O}_{K'}/\mathfrak{m}_{K'}$ and let $e = e_{K'/K}$ be the ramification index. Let $r > 0$ be a rational number and assume that $r' = e'r$ is an integer. We regard the divisor $R' = r'D_{S'} = \mathrm{Spec}\ \mathcal{O}_{K'}/\mathfrak{m}_{K'}^{r'}$ of S' as a closed subscheme of $P_{S'} = P \times_S S'$ by the section $s'\colon S' \to P_{S'}$ induced by $s\colon S \to P$.

We consider the blow-up of $P_{S'}$ at the center R' and let $P_{S'}^{(r)}$ denote the complement of the proper transform of the closed fiber $P_{S'} \times_{S'} D_{S'}$. The scheme $P_{S'}^{(r)}$ is smooth over S' and the closed fiber $P_{S'}^{(r)} \times_{S'} D_{S'}$ is the vector bundle $\Theta_{F'}^{(r)}$ over F' such that the F'-vector space consisting of F'-valued points is canonically identified with $\Omega^1_{X/k}(\log D) \otimes \mathfrak{m}_{K'}^{-r'}/\mathfrak{m}_{K'}^{-r'+1}$.

Example 3.1. *Assume* $X = \mathrm{Spec}\ k[T_1, \ldots, T_d]$ *and* $D = (T_1)$. *Then, we have* $\mathcal{O}_K = k(T_2, \ldots, T_d)[[T_1]]$ *and* $X \times_k S = \mathrm{Spec}\ \mathcal{O}_K[S_1, \ldots, S_d]$. *The canonical map* $S \to X$ *induces a closed immersion* $S \to X \times_k S$ *defined by* $S_i \mapsto T_i$.

The log product $P = X *_k S$ *is* Spec $\mathcal{O}_K[U_1^{\pm 1}, S_2, \dots, S_d]$ *with the canonical map* $P = X *_k S \to X \times_k S$ *defined by* $S_1 \mapsto U_1 T_1$*. If* π' *is a uniformizer of* K'*, the scheme* $P^{(r)}_{S'} = X *_k S$ *is* Spec $\mathcal{O}_{K'}[V_1, \dots, V_d]$ *with the canonical map* $P^{(r)}_{S'} \to P = X *_k S$ *defined by* $U_1 \mapsto 1 + \pi'^{r'} V_1,\ S_i \mapsto 1 + \pi'^{r'} V_i$ *for* $1 < i \leq d$*.*

We consider the normalizations $\bar{Q}^{(r)}_{S'}$ and $\bar{T}_{S'}$ of $Q \times_P P^{(r)}_{S'}$ and of $T \times_S S'$ respectively. Then, the diagram (15) induces a diagram

$$\begin{array}{ccc} \bar{T}_{S'} & \xrightarrow{i^{(r)}} & \bar{Q}^{(r)}_{S'} \\ \downarrow & & \downarrow{\scriptstyle f^{(r)}} \\ S' & \xrightarrow{s^{(r)}} & P^{(r)}_{S'}. \end{array} \tag{16}$$

By the assumption that K' contains L, the scheme $\bar{T}_{S'}$ is isomorphic to the disjoint union of finitely many copies of S' and the geometric fiber $\bar{T}_{\bar{F}} = \bar{T}_{S'} \times_{S'} \bar{F}$ is identified with $\mathrm{Gal}(L/K)$.

3.4. **Definition of the upper ramification groups.** After replacing K' by some finite separable extension, the geometric closed fiber $\bar{Q}^{(r)}_{\bar{F}} = \bar{Q}^{(r)}_{S'} \times_{S'} \mathrm{Spec}\ \bar{F}$ is reduced and the formation of $\bar{Q}^{(r)}_{S'}$ commutes with further base change. We call such $\bar{Q}^{(r)}_{S'}$ a stable integral model. The finite map $i^{(r)} \colon \bar{T}_{S'} \to \bar{Q}^{(r)}_{S'}$ induces surjections

$$\begin{array}{ccc} \bar{T}_{\bar{F}} = \mathrm{Gal}(L/K) & \xrightarrow{i^{(r+)}_*} & f^{(r)-1}(0) \\ & \searrow^{i^{(r)}_*} & \downarrow \\ & & \pi_0(\bar{Q}^{(r)}_{\bar{F}}) \end{array} \tag{17}$$

of finite sets to the set of geometric connected components and to the inverse image of the origin $0 \in P^{(r)}_{\bar{F}} = \Theta^{(r)}_{\bar{F}}$.

Theorem 3.1 ([5, Theorems 3.3, 3.8], [15, Section 1.3])**.** *Let* L *be a finite Galois extension over* K *of Galois group* G *and we consider a diagram* (15) *as above.*

1. *For a rational number* $r > 0$*, we take a finite separable extension* K' *of* K *containing* L *such that* $e_{K'/K} r$ *is an integer and that* $Q^{(r)}_{\bar{S}'}$ *is a stable integral model.*

Then, the inverse image $i^{(r)-1}_*(i^{(r)}_*(1)) = G^r_{\log} \subset G$ *is independent of the choice of diagram* (15) *or an extension* K' *and is a normal subgroup of* G*. Further the surjection* $i^{(r)}_*$ (17) *induces a bijection* $G/G^r_{\log} \to \pi_0(\bar{Q}^{(r)}_{\bar{F}})$*.*

2. *Let the notation be as in* 1. *Then, there exist rational numbers* $0 = r_0 < r_1 < \ldots < r_m$ *such that* $G^r_{\log} = G^{r_i}_{\log}$ *for* $r \in (r_{i-1}, r_i] \cap \mathbb{Q}$ *and* $i = 1, \ldots, m$ *and* $G^r_{\log} = 1$ *for* $r > r_m$.

We put $G^{r+}_{\log} = G^{r_i}_{\log}$ *for* $r \in [r_{i-1}, r_i) \cap \mathbb{Q}$ *and* $i = 1, \ldots, m$ *and* $G^r_{\log} = 1$ *for* $r \geq r_m$. *Then, the surjection* $i^{(r+)}_*$ (17) *induces a bijection* $G/G^{r+}_{\log} \to f^{(r)-1}(0)$.

3. *For a subfield* $M \subset L$ *Galois over* K *and for a rational number* $r > 0$, *the subgroup* $\mathrm{Gal}(M/K)^r_{\log} \subset \mathrm{Gal}(M/K)$ *is the image of* $G^r_{\log} = \mathrm{Gal}(L/K)^r_{\log}$.

The proofs of 1 and 3 are rather straightforward. That of 2 requires some result from rigid geometry.

If L is an abelian extension of K, it is concretely described using the Artin-Schreier-Witt theory as follows.

Example 3.2 ([12], [8])**.** A cyclic extension L of degree p^{m+1} is defined by a Witt vector by the isomorphism $W_{m+1}(K)/(F-1) \to H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ of Artin-Schreier-Witt theory. An increasing filtration on $W_{m+1}(K)$ is defined by

$$F^n W_{m+1}(K) = \{(a_0, \ldots, a_m) \in W_{m+1}(K) \mid p^{m-i} v_K(a_i) \geq -n \text{ for } i = 0, \ldots, m\}.$$

The filtration on $H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ induced by the canonical surjection $W_{m+1}(K) \to H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is considered in [12]. For $G = \mathrm{Gal}(L/K)$, the filtration $(G^n_{\log})_{n\geq 0}$ indexed by integers is the dual of the restriction to $Hom(\mathrm{Gal}(L/K), \mathbb{Z}/p^{m+1}\mathbb{Z}) \subset H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. Namely, we have $G^n_{\log} = \{\sigma \in G \mid c(\sigma) = 0 \text{ if } c \in F^n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})\}$. Further, for a rational number $r \in (n-1, n] \cap \mathbb{Q}$, we have $G^r_{\log} = G^n_{\log}$.

3.5. **Graded pieces.** We study the graded pieces. Let $\Omega^1_{\mathcal{O}_K}(\log)$ denote the free $\mathcal{O}_K$-module $\Omega^1_{X/k}(\log D)_\xi \otimes \mathcal{O}_K$ of rank $\dim X$. By abuse of notation, let $\Omega^1_F(\log)$ denote the F-vector space $\Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} F$. Then, we have an exact sequence $0 \to \Omega^1_F \to \Omega^1_F(\log) \xrightarrow{\mathrm{res}} F \to 0$ of F-vector spaces of finite dimension. We extend the normalized discrete valuation v of K to a separable closure $\bar{K}$ and, for a rational number r, we put $\mathfrak{m}^r_{\bar{K}} = \{a \in \bar{K} \mid v(a) \geq r\}$ and $\mathfrak{m}^{r+}_{\bar{K}} = \{a \in \bar{K} \mid v(a) > r\}$. The $\bar{F}$-vector space $\mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}}$ is of dimension 1.

Corollary ([6, Theorem 2.15], [15, Theorem 1.24, Corollary 1.25])**.** *Let* L *be a finite Galois extension of Galois group* G. *Then, for a rational number* $r > 0$, *the graded quotient* $\mathrm{Gr}^r_{\log} G = G^r_{\log}/G^{r+}_{\log}$ *is abelian and killed by* p.

Further, there exists a canonical injection

$$Hom(\mathrm{Gr}^r_{\log} G, \mathbb{F}_p) \to Hom_{\bar F}(\mathfrak{m}^r_{\bar K}/\mathfrak{m}^{r+}_{\bar K}, \Omega^1_F(\log) \otimes_F \bar F). \tag{18}$$

This is a consequence of the group structure and the étale isogeny proved in Theorem 4.1.

For a non-trivial character $\chi \in Hom(\mathrm{Gr}^r_{\log} G, \mathbb{F}_p)$, we call the image $\mathrm{rsw}\chi \in Hom_{\bar F}(\mathfrak{m}^r_{\bar K}/\mathfrak{m}^{r+}_{\bar K}, \Omega^1_F(\log) \otimes_F \bar F)$ the refined Swan character of χ.

Example 3.3 ([12], [8])**.** We keep the notation in Example 3.2. We define a canonical map $F^m d\colon W_{m+1}(K) \to \Omega^1_K$ by sending $(a_0, \ldots, a_m)$ to $a_0^{p^m-1} da_0 + \cdots + da_m$. It maps $F^n W_{m+1}(K)$ to $F^n\Omega^1_K = \mathfrak{m}_K^{-n}\Omega^1_{\mathcal{O}_K}(\log)$ for $n \in \mathbb{Z}$ and induces an injection

$$\mathrm{Gr}^n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{Gr}^n \Omega^1_K = Hom_F(\mathfrak{m}^n_K/\mathfrak{m}^{n+1}_K, \Omega^1_F(\log)) \tag{19}$$

for $n > 0$.

Let L be a cyclic extension of degree p^{m+1} corresponding to a character $\chi \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. The smallest integer $n \geq 0$ such that $\chi \in F^n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is called the conductor of χ and is equal to the smallest rational number r such that the ramification of L is bounded by $r+$. The character is ramified if and only if the conductor is > 0. For a ramified character χ of conductor $n > 0$, the image of the class of χ by the injection (19) in $Hom_F(\mathfrak{m}^n_K/\mathfrak{m}^{n+1}_K, \Omega^1_F(\log)) \subset Hom_{\bar F}(\mathfrak{m}^n_{\bar K}/\mathfrak{m}^{n+}_{\bar K}, \Omega^1_F(\log) \otimes_F \bar F)$ is the refined Swan character $\mathrm{rsw}\chi$.

4. Wild blow-up and the characteristic class

Let X be a smooth separated scheme of finite type over a perfect field k of characteristic $p > 0$ and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. We consider a finite etale G-torsor V over U for a finite group G and study the ramification of V along D.

4.1. **Wild blow-up and the group structure.**

The ramification of V along D will be measured by linear combinations $R = \sum_i r_i D_i$ with rational coefficients $r_i \geq 0$ of irreducible components of D. In the following, we assume the coefficients of $R = \sum_i r_i D_i$ are integers, for simplicity.

We consider the log product $P = X *_k X \subset (X \times_k X)'$ and the log diagonal $\tilde\delta\colon X \to P = X *_k X$ as in Section 1.1. We define a relatively affine scheme $P^{(R)}$ over P. The scheme $P^{(R)}$ is the complement of the proper transforms of $P \times_X R$ in the blow-up of P at the center $R \subset X$ embedded by the log diagonal map $\tilde\delta\colon X \to P$. The log diagonal map is uniquely lifted to a closed immersion $\delta^{(R)}\colon X \to (X *_k X)^{(R)}$ and the open immersion $U \times U \to X *_k X$ is uniquely lifted to a closed immersion $j^{(R)}\colon U \times U \to (X *_k X)^{(R)}$.

Example 4.1. We take the notation in Example 2.2. If we put $T^R = T_1^{r_1} \cdots T_n^{r_n}$, the scheme $(X *_k X)^{(R)}$ is the spectrum of

$$\begin{aligned} & A[V_1, \ldots , V_d]/(U_1 - 1 - V_1 T^R, \ldots , U_n - 1 - V_n T^R, \\ & \qquad\qquad\qquad S_{n+1} - T_{n+1} - V_{n+1} T^R, \ldots , S_d - T_d - V_d T^R) \\ & = k[T_1, \ldots , T_d, V_1, \ldots , V_d, (1 + V_1 T^R)^{-1}, \ldots , (1 + V_n T^R)^{-1}]. \end{aligned} \tag{20}$$

The immersion $\delta^{(R)}\colon X \to (X *_k X)^{(R)}$ is defined by $V_1 = \cdots = V_d = 0$.

The base change $P^{(R)} \times_X R$ with respect to the projection $P^{(R)} \to X \supset R$ is the twisted tangent bundle $\Theta^{(R)} = \mathbf{V}(\Omega^1_X(\log D)(R)) \times_X R$ where $\mathbf{V}(\Omega^1_X(\log D)(R))$ denotes the vector bundle defined by the symmetric algebra of the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)(R)$.

The projection $\mathrm{pr}_{13}\colon (X \times_k X) \times_k (X \times_k X) = X \times_k X \times_k X \to X \times_k X$ induces a morphism

$$\mu\colon (X *_k X)^{(R)} \times_k (X *_k X)^{(R)} \to (X *_k X)^{(R)}$$

and defines a groupoid structure on $(X *_k X)^{(R)}$. The group structure on the vector bundle $\Theta^{(R)} = P^{(R)} \times_X R$ is compatible with the groupoid structure.

Let V be a G-torsor over U for a finite group G. We consider the quotient $(V \times_k V)/\Delta G$ by the diagonal $\Delta G \subset G \times G$ as a finite etale covering of $U \times_k U$ and let Z be the normalization of $(X *_k X)^{(R)}$ in the quotient $(V \times_k V)/\Delta G$. The diagonal map $V \to V \times_k V$ induces a closed immersion $U = V/G \to (V \times_k V)/\Delta G$ on the quotients and is extended to a closed immersion $e\colon X \to Z$.

Theorem 4.1. *Let X be a separated smooth scheme of finite type over k and $U = X \setminus D$ be the complement of a divisor with simple normal crossings. Let $R = \sum_i r_i D_i \geq 0$ be an effective Cartier divisor.*

Let V be a G-torsor over U for a finite group G. Let X be the normalization of $(X *_k X)^{(R)}$ in the quotient $(V \times_k V)/\Delta G$ and $e\colon X \to Z$ be the section induced by the diagonal.

Assume that Z is etale over $(X *_k X)^{(R)}$ on a neighborhood of the image of $e\colon X \to Z$. Let $Z_0 \subset Z$ be the maximum open subscheme etale over $(X *_k X)^{(R)}$.

1. The base change $Z_{0,R} = Z_0 \times_X R$ with respect to the projection $Z_0 \to (X *_k X)^{(R)} \to X \supset R$ has a natural structure of smooth group scheme over R such that the map $e_R\colon X_R \to Z_{0,R}$ induced by $e\colon X \to Z$ is the unit. Further the etale map $Z_{0,R} \to \Theta^{(R)} = (X *_k X)^{(R)} \times_X R$

induced by the canonical map $Z \to (X *_k X)^{(R)}$ is a group homomorphism.

2. For every point $x \in R$, the connected component $Z^0_{0,x}$ of the fiber $Z_{0,x}$ is isomorphic to the product of finitely many copies of the additive group $\mathbf{G}_{a,x}$ and the map $Z^0_{0,x} \to \Theta^{(R)}_x$ is an etale isogeny.

4.2. **Ramification of a rank 1 sheaf.** As an application, we study the ramification of a rank 1 sheaf. Let $\mathcal{F}$ be a smooth sheaf of rank 1 corresponding to a character $\chi\colon \pi_1(U)^{\rm ab} \to \Lambda^\times$. For each irreducible component D_i, let K_i be the local field and n_i be the conductor of the p-part of the character $\chi_i\colon G^{\rm ab}_{K_i} \to \Lambda^\times$. We put $R = \sum_i n_i D_i$.

We consider a smooth sheaf $\mathcal{H} = \mathcal{H}om({\rm pr}_2^*\mathcal{F}, {\rm pr}_1^*\mathcal{F})$ on $U \times U$. Then, the direct image $j^{(R)}_*\mathcal{H}$ by the open immersion $j^{(R)}\colon U \times_k U \to (X *_k X)^{(R)}$ is a smooth sheaf of Λ-modules of rank 1 on $(X *_k X)^{(R)}$. For a component with $r_i > 0$, the restriction of $j^{(R)}_*\mathcal{H}$ to the fiber $\Theta^{(R)}_{\xi_i}$ is the Artin-Schreier sheaf defined by the refined Swan character ${\rm rsw}\chi_i$ regarded as a linear form on $\Theta^{(R)}_{\xi_i}$ by [7, Proposition 4.2.2].

Further, we assume the following condition:

(C) For each irreducible component D_i of D such that $r_i > 0$, the refined Swan character ${\rm rsw}_i\ \chi$ defines a locally splitting injection

$$\mathrm{rsw}_i\ \chi\colon \mathcal{O}_X(-R) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i}.$$

This condition says that for each irreducible component, the wild ramification of $\mathcal{F}$ is controlled at the generic point. It is called the cleanness condition and studied in [13].

Theorem 4.2 ([7, Theorem 4.2.6])**.** *Assume the condition* (C) *above is satisfied and X is proper. Then, we have*

$$\chi_c(U_{\bar k}, \mathcal{F}) = \deg(X, X)_{(X *_k X)^{(R)}}.$$

It is proved by showing that the characteristic class introduced in the following equals the intersection product $(X, X)_{(X *_k X)^{(R)}}$. A generalization to higher rank case is studied in [15, Theorem 3.4].

4.3. **Characteristic class.** Let X be a separated scheme of finite type over a field k. As a coefficient ring Λ, we consider a ring finite over $\mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ for a prime number $\ell \neq {\rm char}\, k$. Let $a\colon X \to {\rm Spec}\, k$ denote the structure map and $K_X = Ra^!\Lambda$ denote the dualizing complex. If X is smooth of dimension d over k, we have $K_X = \Lambda(d)[2d]$.

Let $\mathcal{F}$ be a constructible sheaf of flat Λ-modules on X and consider the object

$$\mathcal{H} = R\mathcal{H}om({\rm pr}_2^*\mathcal{F}, R{\rm pr}_1^!\mathcal{F})$$

of the derived category $D_{\rm ctf}(X \times_k X, \Lambda)$ of constructible sheaves of Λ-modules of finite tor-dimension on the product $X \times_k X$. If X is smooth of dimension d over k and if $\mathcal{F}$ is smooth, we have a canonical isomorphism $\mathcal{H} \to \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})(d)[2d]$.

A canonical isomorphism

$$End(\mathcal{F}) \to H^0_X(X \times_k X, \mathcal{H}) \tag{21}$$

is defined in [10]. Hence, we may regard the identity $\mathrm{id}_{\mathcal{F}}$ as a cohomology class $\mathrm{id}_{\mathcal{F}} \in H^0_X(X \times_k X, \mathcal{H})$ supported on the diagonal $X \subset X \times_k X$. Let $\delta\colon X \to X \times_k X$ be the diagonal map. Further in [10], a canonical map $\delta^*\mathcal{H} \to K_X$ is defined as the trace map. The characteristic class

$$C(\mathcal{F}) \in H^0(X, K_X)$$

is defined as the image of the pull-back $\delta^*\mathrm{id}_{\mathcal{F}} \in H^0(X, \delta^*\mathcal{H})$ by the induced map $H^0(X, \delta^*\mathcal{H}) \to H^0(X, K_X)$. If X is smooth and if $\mathcal{F}$ is smooth, we have $C(\mathcal{F}) = \mathrm{rank}\ \mathcal{F} \cdot (X, X)_{X \times_k X}$ where $(X, X)_{X \times_k X}$ denotes the self-intersection in the product $X \times_k X$. The Lefschetz trace formula [10] asserts that, if X is proper, the trace map $H^0(X, K_X) \to \Lambda$ sends the characteristic class $C(\mathcal{F})$ to the Euler number $\chi(X_{\bar{k}}, \mathcal{F})$. In other words, the characteristic class is a geometric refinement of the Euler number.

A relation of the canonical class with the Swan class is given in [7, Theorem 3.3.1].

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