Project description

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0 Outline

The Grothendieck-Ogg-Shafarevich (GOS) formula (*cf.* [SGA5, Exposé X]) is one of the most classical results in geometric ramification theory. It describes the Euler characteristic of a smooth ℓ-adic sheaf $\mathcal{F}$ on a smooth curve over an algebraically closed field in terms of a local ramification invariant of $\mathcal{F}$ called the Swan conductor. In [KS08], Kato and Saito obtained a higher-dimensional generalization of it. The first aim of this project is to find some concrete examples of their rather abstract formula (for a detailed plan, see Section 1).

The second and main theme of our project is to consider an arithmetic variant of the GOS formula. Let V be a complete discrete valuation ring with algebraically closed residue field. Put $S = \operatorname{Spec} V$ and consider a scheme X separated of finite type over S whose generic fiber X_η is smooth. For a smooth ℓ-adic étale sheaf $\mathcal{F}$ on X_η, we can attach the nearby cycle complex $R\psi\mathcal{F}$, which is an object of the derived category $D^b_c(X_s, \mathbb{Q}_\ell)$ of ℓ-adic sheaves over the special fiber of X (*cf.* [SGA7, Exposé XIII]). In our project, we will study the compactly supported cohomology $H^i_c(X_s, R\psi\mathcal{F})$, especially its Euler characteristic $\chi_c(X_s, R\psi\mathcal{F}) = \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^i_c(X_s, R\psi\mathcal{F})$. If X is proper over S, the proper base change theorem tells us that $H^i_c(X_s, R\psi\mathcal{F})$ is isomorphic to $H^i(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}})$, the ℓ-adic cohomology of the generic fiber. In this case, no contribution of ramification of $\mathcal{F}$ appears. However, if X is not proper over S, the nearby cycle cohomology $H^i_c(X_s, R\psi\mathcal{F})$ should reflect arithmetic ramification of $\mathcal{F}$. Note that $H^i_c(X_s, R\psi\mathcal{F})$ is not necessarily isomorphic to $H^i_c(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}})$. The Euler characteristic $\chi_c(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}})$ of the latter can be described by the classical GOS formula, and only reflects ramification of $\mathcal{F}$ on the generic fiber.

In the language of rigid geometry, the nearby cycle cohomology $H^i_c(X_s, R\psi\mathcal{F})$ can be interpreted as follows. Let $X^\wedge$ be the formal completion of X along X_s and $\mathfrak{X}$ the rigid generic fiber of $X^\wedge$ in the sense of Raynaud. Then, $H^i_c(X_s, R\psi\mathcal{F})$ is naturally isomorphic to the étale cohomology $H^i_c(\mathfrak{X}_{\overline{\eta}}, \mathcal{F}^{\mathrm{rig}}_{\overline{\eta}})$ of the rigid space $\mathfrak{X}$ with a naturally induced coefficient. In this context, when $\dim X_\eta = 1$, Huber [Hub01] proved a formula of GOS type[1]:

$$\chi_c(X_s, R\psi\mathcal{F}) = \chi_c(\mathfrak{X}_{\overline{\eta}}, \mathcal{F}^{\mathrm{rig}}_{\overline{\eta}}) = \operatorname{rank}\mathcal{F}\cdot\chi_c(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_\ell) - \sum_{x\in\mathfrak{X}^c_{\overline{\eta}}\setminus\mathfrak{X}_{\overline{\eta}}} \operatorname{Sw}_x\mathcal{F}.$$

[1]Precisely speaking, Huber considered a locally constant torsion sheaf $\mathcal{F}$.

Here $\mathfrak{X}^c_{\overline{\eta}}$ is the so-called universal compactification of $\mathfrak{X}_{\overline{\eta}}$. To a point x on the boundary $\mathfrak{X}^c_{\overline{\eta}} \setminus \mathfrak{X}_{\overline{\eta}}$ of the universal compactification, a valuation field $\kappa(x)$ of rank 2 is naturally attached, and a local ramification invariant $\mathrm{Sw}_x \mathcal{F}$ at x, an analogue of the Swan conductor, is defined in a similar way as in the classical case (for example, we can use the natural ramification filtration on the Galois group of $\kappa(x)$). This formula provides a powerful method to study the cohomology of some arithmetic curves, such as the Lubin-Tate tower for GL(2) (*cf.* [Wew05]).

In this project, we will try to understand the formula above in the style of Kato-Saito's ramification theory, and generalize it to a higher-dimensional case. For a detail, see Section 2.

1 The case over a field

Throughout this section, let k be an algebraically closed field with characteristic $p > 0$. Fix a prime number ℓ different from p.

1.1 Warming up: one-dimensional case

Let X be an irreducible proper smooth curve over k and U a non-empty open subset of X. To a smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U and a point $x \in X \setminus U$, we can attach a ramification invariant $\mathrm{Sw}_x(\mathcal{F}) \in \mathbb{Z}$ called the Swan conductor (for the definition, see Saito's lecture notes). By using this invariant, the Euler characteristic $\chi_c(U, \mathcal{F}) = \sum_{i=0}^{2} (-1)^i \dim_{\overline{\mathbb{Q}}_\ell} H^i_c(U, \mathcal{F})$ can be computed as follows:

Theorem 1.1 (The Grothendieck-Ogg-Shafarevich formula)

$$\chi_c(U, \mathcal{F}) = \mathrm{rank}(\mathcal{F}) \cdot \chi_c(U, \overline{\mathbb{Q}}_\ell) - \sum_{x \in X \setminus U} \mathrm{Sw}_x(\mathcal{F}).$$

Problem 1.2 Fix an integer $m \geq 1$ which is prime to p. We have an étale covering of $\mathbb{G}_m = \mathrm{Spec}\, k[T, T^{-1}]$ defined by the equation $S^m = T$ (the Kummer covering). It is a Galois covering with Galois group $\mathbb{Z}/m\mathbb{Z}$. Therefore, a non-trivial character $\chi\colon \mathbb{Z}/m\mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ gives rises to a smooth sheaf $\mathcal{L}_\chi$ of rank 1 over $\mathbb{G}_m$.

i) Consider the compactification $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$ of $\mathbb{G}_m$ (namely, consider the case where $X = \mathbb{P}^1$ and $U = \mathbb{G}_m$). For the points $0, \infty \in \mathbb{P}^1 \setminus \mathbb{A}^1$, compute the Swan conductors $\mathrm{Sw}_0(\mathcal{L}_\chi)$, $\mathrm{Sw}_\infty(\mathcal{L}_\chi)$.

ii) By using the GOS formula, compute the Euler characteristic $\chi_c(\mathbb{G}_m, \mathcal{L}_\chi)$. Observe that $H^i_c(\mathbb{G}_m, \mathcal{L}_\chi) = 0$ for $i \neq 1$, and determine the dimension of $H^1_c(\mathbb{G}_m, \mathcal{L}_\chi)$.

Problem 1.3 We have an étale covering of $\mathbb{A}^1 = \mathrm{Spec}\, k[T]$ defined by the equation $S^p - S = T$ (the Artin-Schreier covering). It is a Galois covering with Galois group $\mathbb{Z}/p\mathbb{Z}$. Therefore, a non-trivial character $\psi\colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ gives rises to a smooth sheaf $\mathcal{L}_\psi$ of rank 1 over $\mathbb{A}^1$.

i) Consider the compactification $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ of $\mathbb{A}^1$ (namely, consider the case where $X = \mathbb{P}^1$ and $U = \mathbb{A}^1$). For the point $\infty \in \mathbb{P}^1 \setminus \mathbb{A}^1$, compute the Swan conductor $\mathrm{Sw}_\infty(\mathcal{F})$.

ii) By using the GOS formula, compute the Euler characteristic $\chi_c(\mathbb{A}^1, \mathcal{L}_\psi)$. Observe that $H^i_c(\mathbb{A}^1, \mathcal{L}_\psi) = 0$ for $i \neq 1$, and determine the dimension of $H^1_c(\mathbb{A}^1, \mathcal{L}_\psi)$.

iii) More generally, for $f \in k[T]$, we can consider the smooth sheaf of rank 1 over $\mathbb{A}^1$ associated with the étale covering defined by $S^p - S = f$. If we identify f with the morphism $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1$, then this sheaf is nothing but the pull-back $f^*\mathcal{L}_\psi$. Can you compute $\mathrm{Sw}_\infty(f^*\mathcal{L}_\psi)$ and $\chi_c(\mathbb{A}^1, f^*\mathcal{L}_\psi)$?

Problem 1.4 (*cf.* [SGA4$\frac{1}{2}$, Sommes trig.]) Here we assume that $k = \overline{\mathbb{F}}_q$, where $q = p^n$ is a power of p. Fix a non-trivial multiplicative (resp. additive) character $\chi\colon \mathbb{F}_q^\times \longrightarrow \overline{\mathbb{Q}}_\ell$ (resp. $\psi\colon \mathbb{F}_q \longrightarrow \overline{\mathbb{Q}}_\ell$). Then, the étale covering $\mathbb{G}_{m,\mathbb{F}_q} \longrightarrow \mathbb{G}_{m,\mathbb{F}_q}$; $x \longmapsto x^{q-1}$ and χ give a smooth sheaf $\mathcal{L}_\chi$ of rank 1 over $\mathbb{G}_{m,\mathbb{F}_q} = \operatorname{Spec} \mathbb{F}_q[T, T^{-1}]$. Similarly, the étale covering $\mathbb{A}^1_{\mathbb{F}_q} \longrightarrow \mathbb{A}^1_{\mathbb{F}_q}$; $x \longmapsto x^q - x$ and ψ give a smooth sheaf $\mathcal{L}_\psi$ of rank 1 over $\mathbb{A}^1_{\mathbb{F}_q} = \operatorname{Spec} \mathbb{F}_q[T]$. We denote the pull-back of $\mathcal{L}_\chi$ and $\mathcal{L}_\psi$ to $\mathbb{G}_m = \operatorname{Spec} k[T, T^{-1}]$ by the same symbols.

i) Compute $\chi_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ and $\dim_{\overline{\mathbb{Q}}_\ell} H^1_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ by the GOS formula.

ii) Use the Grothendieck-Lefschetz trace formula (*cf.* [SGA4$\frac{1}{2}$, Rapport, Théorème 3.2]) to prove the following:

$$G(\chi, \psi) := \sum_{a \in \mathbb{F}_q^\times} \psi(a)\chi(a) = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}\bigl(\mathrm{Frob}_q; H^i_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)\bigr).$$

The left hand side is called the Gauss sum.

iii) Observe that the natural map $H^1_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi) \longrightarrow H^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ is an isomorphism. By this and the Weil conjecture, we may conclude that every eigenvalue α of Frob_q on $H^i_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ satisfies $|\iota(\alpha)| = q^{1/2}$ for an arbitrary isomorphism of fields $\iota\colon \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$. Together with i), we obtain a well-known identity $|\iota G(\chi, \psi)| = q^{1/2}$.

1.2 Higher-dimensional case

In [KS08], the GOS formula has been generalized to the higher-dimensional case. We would like to give some concrete examples of their theory. Here we consider $\overline{\mathbb{F}}_\ell$-sheaves instead of $\overline{\mathbb{Q}}_\ell$-sheaves. In fact, the definition of the Swan conductor for $\overline{\mathbb{Q}}_\ell$-sheaves in [KS08] is reduced to that for $\overline{\mathbb{F}}_\ell$-sheaves. Fix a non-trivial character $\psi\colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \overline{\mathbb{F}}_\ell^\times$.

Put $U = \mathbb{G}_m^2 = \operatorname{Spec} k[S, T, S^{-1}, T^{-1}]$. For $a, b \in \mathbb{Z}$, consider the morphism $f_{a,b}\colon U \longrightarrow \mathbb{A}^1$; $(x, y) \longmapsto x^a y^b$. We want to compute $\chi_c(U, f_{a,b}^*\mathcal{L}_\psi)$ by the GOS formula (the definition of $\mathcal{L}_\psi$ is the same as in Problem 1.3).

Problem 1.5 Consider the case where $(a,b)=(1,-1)$. Write $f=f_{1,-1}$.

i) Let X be the blow-up of $\mathbb{P}^2$ at the origin. Then $f\colon U \longrightarrow \mathbb{A}^1$ extends to $\overline{f}\colon X \longrightarrow \mathbb{P}^1$, which gives a compactification of f. Compute the Swan divisor $\mathrm{sw}(f^*\mathcal{L}_\psi)$ of $f^*\mathcal{L}_\psi$ and the refined Swan conductor with respect to this compactification (*cf.* [Kat89], [Kat94], [KS08, §5]. In particular, [Kat94, (3.6)] will be useful for our calculation). Observe that $f^*\mathcal{L}_\psi$ is clean with respect to $U\subset X$.

ii) Use the GOS formula to obtain $\chi_c(U,f^*\mathcal{L}_\psi)$.

iii) Compare the result in ii) with Laumon's GOS formula ([Lau83, Théorème 1.2.1]).

Problem 1.6 Next consider the case where $(a,b)=(1,-2)$. Write $f=f_{1,-2}$.

i) Let W be the blow-up of $\mathbb{A}^2=\operatorname{Spec} k[S,T]$ along the ideal (S,T^2). Since W is not smooth over k, we take a toric resolution $\widetilde{W}$ of W. In this case, $\widetilde{W}$ is the blow-up of W along the strict transform of the ideal (S,T). Then $f\colon U \longrightarrow \mathbb{A}^1$ extends to $\overline{f}\colon \widetilde{W} \longrightarrow \mathbb{P}^1$, which gives a partial compactification of f. Compute the Swan divisor $\mathrm{sw}(f^*\mathcal{L}_\psi)$ of $f^*\mathcal{L}_\psi$ and the refined Swan conductor with respect to this partial compactification. Is $f^*\mathcal{L}_\psi$ clean with respect to $\widetilde{W}$?

ii) Find a nice compactification $U \hookrightarrow X$ so that $f^*\mathcal{L}_\psi$ is clean with respect to it. Compute the Swan divisor.

iii) Use the GOS formula to obtain $\chi_c(U,f^*\mathcal{L}_\psi)$. To compute the intersection number, it will be convenient to use the intersection theory on toric varieties (*cf.* [Ful93, Chapter 5]).

Problem 1.7 Extend the arguments in Problem 1.5 and Problem 1.6 to $f_{a,b}^*\mathcal{L}_\psi$ for general (a,b). Is it possible to apply a similar method to the case of dimension greater than 2?

Problem 1.8 A similar but slightly different case is considered by Denef and Loeser ([DL91]). Can we recover their result [DL91, Theorem 1.2] by using the GOS formula (at least in the 2-dimensional case)?

2 Arithmetic case

In this section, let K be a complete discrete valuation field of mixed characteristic $(0,p)$. Denote by $\mathcal{O}_K$ the ring of integers of K. We write v_K for the normalized valuation $K^\times \twoheadrightarrow \mathbb{Z}$. Assume that the residue field k of $\mathcal{O}_K$ is algebraically closed.

Let us recall Huber's GOS formula for rigid curves ([Hub01]). Here we use terminology in rigid geometry. Let $\mathfrak{U}$ be a quasi-compact rigid curve which is separated and smooth over K, and $\mathcal{F}$ a smooth $\overline{\mathbb{F}}_\ell$-sheaf on $\mathfrak{U}$. We have the universal compactification $\mathfrak{U}^c$ of $\mathfrak{U}$ (*cf.* [Hub01, 5.9, Example 5.10]). For each $x\in\mathfrak{U}^c\setminus\mathfrak{U}$, a

2-dimensional henselian valuation field $\kappa(x)$ with valuation $v_x\colon \kappa(x)^\times \twoheadrightarrow \Gamma_x$ is naturally attached. The field $\kappa(x)$ is an extension field of K and we have the following commutative diagram:

$$\begin{array}{ccccc} K^\times & & \xrightarrow{\quad v_K \quad} & & \mathbb{Z} \\ \big\downarrow & & & & \big\downarrow{\scriptstyle m\mapsto(m,0)} \\ \kappa(x)^\times & \xrightarrow{v_x} & \Gamma_x & \underset{(*)}{\overset{\cong}{\longleftrightarrow}} & \mathbb{Z}\times\mathbb{Z}, \end{array}$$

where $\mathbb{Z}\times\mathbb{Z}$ is considered as a totally ordered commutative group by the lexicographic order, and $(*)$ is an ordered isomorphism (in fact, $(*)$ is uniquely determined by the commutative diagram above).

By using ramification theory for $G_{\kappa(x)} = \mathrm{Gal}(\kappa(x)^{\mathrm{sep}}/\kappa(x))$, we can define the Swan conductor $\mathrm{Sw}_x(\mathcal{F}) \in \mathbb{Z}$. As in the classical GOS formula, this ramification-theoretic invariant is related to the Euler characteristic $\chi_c(\mathfrak{U}_{\overline{K}}, \mathcal{F})$:

$$\chi_c(\mathfrak{U}_{\overline{K}}, \mathcal{F}) = \mathrm{rank}(\mathcal{F})\cdot\chi_c(\mathfrak{U}_{\overline{K}}, \overline{\mathbb{F}}_\ell) - \sum_{x\in\mathfrak{U}^c\setminus\mathfrak{U}} \mathrm{Sw}_x(\mathcal{F}).$$

See [Hub01, Theorem 10.2]. Here we call it Huber's GOS formula.

Recall the definition of $\mathrm{Sw}_x(\mathcal{F})$. Let $j\colon \mathfrak{U} \hookrightarrow \mathfrak{U}^c$ be the natural open immersion. Then, the stalk of $j_*\mathcal{F}$ at x gives an $\overline{\mathbb{F}}_\ell$-representation $\rho_{\mathcal{F},x}$ of $G_{\kappa(x)}$. Take a finite Galois extension $L/\kappa(x)$ such that $\rho_{\mathcal{F},x}$ factors through $\mathrm{Gal}(L/\kappa(x))$. The valuation v_x canonically extends to a valuation $v_x\colon L^\times \twoheadrightarrow \Gamma_L$ on L, where $\Gamma_L \subset \Gamma_x\otimes_{\mathbb{Z}}\mathbb{Q}$ is again naturally isomorphic to $\mathbb{Z}\times\mathbb{Z}$.

Definition 2.1 Let γ_L be the minimal element in $\{\gamma\in\Gamma_L \mid \gamma > 0\}$ (it corresponds to $(0,1)\in\mathbb{Z}\times\mathbb{Z}$). Let $\#\colon \Gamma_L \longrightarrow \mathbb{Z}$ be the map $\Gamma_L \cong \mathbb{Z}\times\mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}$. It is characterized by $\# v_K(a) = 0$ for $a\in K^\times$ and $\#\gamma_L = 1$.

For $\sigma\in\mathrm{Gal}(L/\kappa(x))$ with $\sigma\neq 1$, put

$$h_{L/\kappa(x)}(\sigma) = \min\Big\{ v_x\Big(\frac{\sigma(a)-a}{a}\Big) \;\Big|\; a\in L^\times \Big\} \in \Gamma_x.$$

It is known that if $\varpi_L\in L^\times$ satisfies $v_x(\varpi_L) = \gamma_L$, then

$$h_{L/\kappa(x)}(\sigma) = v_x\Big(\frac{\sigma(\varpi_L)-\varpi_L}{\varpi_L}\Big).$$

Define $s_{L/\kappa(x)}\colon \mathrm{Gal}(L/\kappa(x)) \longrightarrow \mathbb{Z}$ by

$$s_{L/\kappa(x)}(\sigma) = -\# h_{L/\kappa(x)}(\sigma)\ (\sigma\neq 1), \quad s_{L/\kappa(x)}(1) = \sum_{\sigma\neq 1} \# h_{L/\kappa(x)}(\sigma).$$

Then, there exists a virtual representation $\mathrm{Sw}_{L/\kappa(x)}$ of $\mathrm{Gal}(L/\kappa(x))$ over $\mathbb{Z}_\ell$ whose character coincides with $s_{L/\kappa(x)}$ (*cf.* [Hub01, Theorem 4.1]). Now we put

$$\mathrm{Sw}_x(\mathcal{F}) = \mathrm{Sw}_{\kappa(x)}(\rho_{\mathcal{F},x}) = \dim_{\overline{\mathbb{F}}_\ell} \mathrm{Hom}_{\mathrm{Gal}(L/\kappa(x))}(\mathrm{Sw}_{L/\kappa(x)}\otimes_{\mathbb{Z}_\ell}\overline{\mathbb{F}}_\ell, \rho_{\mathcal{F},x}).$$

Remark 2.2 We can also define the Swan conductor by using the upper ramification filtration of $G_{\kappa(x)}$ (*cf.* [Hub01, §2]), as in the classical case.

Huber's GOS formula can also be formulated within the framework of scheme theory. For given rigid curve $\mathfrak{U}$, we can always find

- a regular scheme X which is proper, flat and relatively one-dimensional over $\mathcal{O}_K$,
- and an open subscheme U of X such that $X_s \setminus U_s$ consists of finite points and the special fiber X_s is smooth at every point in $X_s \setminus U_s$

such that the rigid generic fiber of the formal completion of U along U_s is isomorphic to $\mathfrak{U}$. Assume moreover that our $\overline{\mathbb{F}}_\ell$-sheaf $\mathcal{F}$ on $\mathfrak{U}$ is induced from a smooth $\overline{\mathbb{F}}_\ell$-sheaf, also denoted by $\mathcal{F}$, on the generic fiber U_η of U. Then,

- we have $\chi_c(\mathfrak{U}_{\overline{K}}, \mathcal{F}) = \chi_c(U_s, R\psi\mathcal{F})$, and
- there exists a canonical bijection $\mathfrak{U}^c \setminus \mathfrak{U} \cong X_s \setminus U_s$.

For $x \in \mathfrak{U}^c \setminus \mathfrak{U} = X_s \setminus U_s$, the 2-dimensional valuation field $(\kappa(x), v_x)$ can be described as follows. Let Y be the unique irreducible component of X_s containing x and ξ the generic point of Y. Then, ξ gives a prime ideal $\mathfrak{p}$ of the henselian local ring $A = \mathcal{O}^h_{X,x}$. Let V_x be the inverse image of $A/\mathfrak{p}$ under the canonical surjection $A_\mathfrak{p} \longrightarrow A_\mathfrak{p}/\mathfrak{p} = \operatorname{Frac} A/\mathfrak{p}$. It is a two-dimensional valuation ring; indeed, $A_\mathfrak{p}$ and $A/\mathfrak{p}$ are discrete valuation rings, and the construction above is a well-known process to compose two valuation rings. For an arbitrary lift $\widetilde{t} \in \mathcal{O}_{X,x}$ of a local coordinate t of Y at x, the valuation $v_x\colon \operatorname{Frac} A_\mathfrak{p} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ corresponding to V_x can be characterized by the formula $v_x(a\widetilde{t^m}) = (v_K(a), m)$, where $a \in K$ and $m \in \mathbb{Z}_{\geq 0}$. We can identify $(\operatorname{Frac} A_\mathfrak{p}, v_x)$ with the valuation field $(\kappa(x), v_x)$.[2] The representation $\rho_{\mathcal{F},x}$ of $G_{\kappa(x)}$ is the pull-back of the $\pi_1(U_\eta)$-representation attached to $\mathcal{F}$ under the natural homomorphism $G_{\kappa(x)} = \pi_1(\operatorname{Frac} A_\mathfrak{p}) \longrightarrow \pi_1(U)$.

Problem 2.3 Consider the case where $X = \mathbb{P}^1_{\mathcal{O}_K}$ and $U = \operatorname{Spec} \mathcal{O}_K[T, T^{-1}]$. For an integer $m \geq 0$, we have a covering of U defined by the equation $S^m = T$, which is étale over the generic fiber. This covering and a non-trivial character $\chi\colon \mathbb{Z}/m\mathbb{Z} \longrightarrow \overline{\mathbb{F}}_\ell^\times$ give rises to a smooth sheaf $\mathcal{L}_\chi$ of rank 1 over U_η. Compute $\chi_c(U_s, R\psi\mathcal{L}_\chi)$ by using Huber's GOS formula. More generally, consider the case of a covering defined by the equation $S^m = f$, where $f \in \mathcal{O}_K[T]$.

Problem 2.4 Assume that K contains a primitive p-th root of unity ζ_p. We put $z = \zeta_p - 1$ and consider a covering of $\mathbb{A}^1_{\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[T]$ defined by the equation

$$\frac{(1 - zS)^p - 1}{z^p} = -f,$$

where $f \in \mathcal{O}_K[T]$ (*cf.* [KS10, §8.1]). On the generic fiber, it is a Kummer covering of degree p; on the other hand, it induces an Artin-Schreier covering $T^p - T = \overline{f}$ on

[2] Actually they are slightly different (we need certain completion of $\operatorname{Frac} A_\mathfrak{p}$), but it does not affect computations of the Swan conductor.

the special fiber. Let $U \subset \mathbb{A}^1_{\mathcal{O}_K}$ be the étale locus of this covering, and $X = \mathbb{P}^1_{\mathcal{O}_K}$. This covering and a non-trivial character $\psi\colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \overline{\mathbb{F}}_\ell^\times$ give a smooth sheaf $\mathcal{L}_{f,\psi}$ of rank 1 over U_η. Compute $\chi_c(U_s, R\psi\mathcal{L}_{f,\psi})$ by using Huber's GOS formula.

If the rank of $\mathcal{F}$ is one, $\mathrm{Sw}_x(\mathcal{F})$ can be described by means of the class field theory for higher dimensional local fields (*cf.* [Kat87, §4]). This suggests us that $\mathrm{Sw}_x(\mathcal{F})$ is related to the refined Swan conductor in [Kat89] and [Kat94].

Problem 2.5 In the cases of Problem 2.3 and Problem 2.4, compute the refined Swan conductor of $\mathcal{L}_\chi$ and $\mathcal{L}_{f,\psi}$, respectively (to do it, [Kat89, §4] and [KS10, §8] will be useful). Find a relationship between Sw_x and the refined Swan conductor.

Problem 2.6 Consider a higher-dimensional generalization of Huber's GOS formula in the rank 1 case by using the refined Swan conductor.

Problem 2.7 A proof of Huber's GOS formula is based on the Lefschetz trace formula [Hub01, Theorem 6.3]. Instead of this formula, use the log Lefschetz trace formula of Kato-Saito [KS10, §1] to obtain another GOS formula. Note that, in general if X is a proper strictly semistable $\mathcal{O}_K$-scheme and U is an open subscheme of X such that $H = X \setminus U$ is flat over $\mathcal{O}_K$ and $(X, X_s \cup H)$ is a strictly semistable pair (*cf.* [dJ96, 6.3]), then the nearby cycle cohomology $H^i_c(U_s, R\psi\mathbb{Q}_\ell)$ is isomorphic to $H^i_c(U_{\overline{K}}, \mathbb{Q}_\ell)$. This makes it possible to apply the log Lefschetz formula to $H^i_c(U_s, R\psi\mathbb{Q}_\ell)$.

Problem 2.8 Try to find a higher-dimensional generalization of the formula developed in Problem 2.7 by using a similar method as in [KS08]. Compare it with the rank 1 case considered in Problem 2.6.

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