Note: In this document, unless otherwise noted, all rings are assumed to be commutative with 1, and all ring homomorphisms take 1 to 1. The sources [HH], [BW], [CS], and [MS] are the notes of Harbater-Hartmann, Bouw-Wewers, Chinburg-Stover, and Mieda-Saito, respectively.

Problem Set 1: Local Fields

Recall that a *discrete valuation ring* (DVR) is a Noetherian, integrally closed, local domain of dimension 1. One can show that such a ring has a unique nonzero ideal. This is a maximal ideal, and is principal. If A is a DVR and m is the maximal ideal, then A/m is the *residue field* of A. Any generator of m is called a *uniformizer* of A. If $K = \mathrm{Frac}(A)$ for a DVR A, then K is called a discrete valuation field (DVF). Note that, essentially by definition, the localization of a Dedekind domain at any maximal ideal is a DVR.

Let A be a DVR with fraction field K and uniformizer π. Then K comes with a *valuation* homomorphism

$$v : K^\times \to \mathbb{Z}$$

given by $v(x) = i$ exactly when $x = u\pi^i$, with u a unit of A.

Problem 1. Show that the valuation v satisfies the *ultrametric inequality*, that is, $v(x+y) \leq \max(v(x), v(y))$. Show that equality holds when $v(x) \neq v(y)$.

Problem 2. Let $c \in (0, \infty)$, and define an absolute value on $K^\times$ by $|0| = 0$ and $|x| = c^{-v(x)}$ when $x \neq 0$. Show that this defines a metric on K, and thus a topology. Show that the topology is independent of c. Show that K is a topological field under this topology (i.e., addition, multiplication, and inversion are continuous). What is the unit disk centered at 0?

Problem 3. For $a \in A$ and $r \in (0, \infty)$, let $B(a, r)$ be the open disk of radius r centered at a. Give meaning to the sentence "any element of the disk $B(a, r)$ is its center." Prove your statement.

If π generates the maximal ideal m of A, then the topology above is called the *m-adic topology* or the *π-adic topology*. When we speak of a DVR or DVF as a topological ring, it is assumed that we mean this one.

A *complete DVR* is a DVR that is complete as a topological space. If A is a DVR and m is its maximal ideal, then the *completion* of A is

$$\hat{A} := \varprojlim_n A/m^n.$$

It is a complete DVR (you can prove this if you want).

Problem 4. Show that the canonical homomorphism $A \to \hat{A}$ is injective, that the image of a uniformizer of A is a uniformizer of $\hat{A}$, and that the residue field of A is the same as that of $\hat{A}$. *Note:* If A was already complete, then the canonical injection is an isomorphism.

Problem 5. (Hensel's Lemma) Let $\hat{A}$ be a complete DVR with maximal ideal m and residue field k. Let f be a monic polynomial with coefficients in $\hat{A}$ and let $\overline{f}$ be its reduction mod m. Show that if $\overline{f}(x) = 0$ has a solution $\overline{a} \in k$ such that $\overline{f}'(\overline{a}) \neq 0$, then there exists $a \in A$ such that $f(a) = 0$ and the reduction of $a \pmod{m}$ is $\overline{a}$. *Hint: Newton's method*

For any prime p, define the *p-adic numbers* $\mathbb{Z}_p$ to be the completion of the local ring $\mathbb{Z}_{(p)}$. Let $\mathbb{Q}_p = \text{Frac}(\mathbb{Z}_p)$.

Problem 6. Characterize the squares in $\mathbb{Z}_p$. *Hint:* For $p = 2$, the binomial theorem may help.

Problem 7. Show that for $p \neq q$, $\mathbb{Q}_p \not\cong \mathbb{Q}_q$.

For any field k, the ring $k[[t]]$ is a complete DVR with residue field k and fraction field $k((t))$. Since the characteristic of the DVR is the same as the residue field, this is called a complete DVR of *equal characteristic.* As a matter of fact, *every* complete DVR in equal characteristic is of the form $k[[t]]$.

Problem 8. If k is of characteristic p, show that every inseparable extension of $k((t))$ is given by adjoining a qth root of t, where q is a power of p.

If k is a perfect field of characteristic p, then there is a unique complete DVR of characteristic 0 with residue field k, such that p is a uniformizer. Since the characteristic of the DVR is not the same as the characteristic of the residue field, this is called a DVR of *mixed characteristic.* This is called the ring of *Witt vectors* over k, and written $W(k)$. The construction can be found in Serre's Local Fields and Lang's Algebra.

Problem 9. What is $W(\mathbb{F}_p)$?

Problem 10. Show that, if r is prime to p, then there exists a field k of characteristic p such that $W(k)$ contains a primitive rth root of unity.

Problem 11. Does there exist a field k of characteristic p such that $W(k)$ contains a pth root of unity?

Note that the term *local field* generally refers to a complete DVF with finite residue field. Such fields are always finite extensions of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, for some p.

Problem Set 2: Extensions of Complete DVR's and Ramification

Let A be a complete DVR and K its fraction field, and assume for the rest of this problem set that the residue field k of A is *perfect*. If L/K is a finite field extension, then L has a natural topology as a K-vector space, and it is complete for this topology. Taking B to be the integral closure of A in L, then B is a DVR with fraction field L, and its natural topology coincides with the topology inherited from L. Thus B is a complete DVR. Also, this shows that if σ is any automorphism of B fixing A, then σ preserves the valuation on B.

The residue field ℓ of B is a finite extension of the residue field k of A. The rank $[\ell : k]$ is called the *residual degree* of the extension, and is sometimes written $f_{L/K}$. Also, if π_A is a uniformizer of A, then π_A is an element of B. The valuation of π_A (considered as an element of B) is called the *inertial degree* of L/K, and is sometimes written $e_{L/K}$. One can show (and you may use) that $e_{L/K} f_{L/K} = [L : K]$. If A is a DVR of mixed characteristic (with residue characteristic p), the *absolute ramification index* of A is the valuation of p.

Problem 1. Let $K = \mathbb{Q}_7$. Compute the residual and inertial degrees of $K(\sqrt{5})/K$, $K(\sqrt{14})/K$, and $K(\sqrt{70})/K$.

Let L/K be a G-Galois extension, for some finite group G. We know that G preserves the valuation on L. Thus the action of G on L descends to an action of G on ℓ.

Problem 2. Show that if L/K is unramified (i.e., $e_{L/K} = 1$) iff the action of G on ℓ is faithful.

Problem 3. Show that if L/K is totally ramified (i.e., $e_{L/K} = |G|$) iff the action of G on ℓ is trivial.

Problem 4. Let L/K be a G-Galois extension as above. Show that there exists a unique subextension M/K of L/K that is unramified, such that L/M is totally ramified.

In the situation above, M/K is called the *maximal unramified subextension* of L/K, and the group $G_0 := \mathrm{Gal}(L/M) \subseteq \mathrm{Gal}(L/K) = G$ is called the *inertia group* of the extension L/K. Clearly, one has $|G_0| = e_{L/K}$.

In fact, one can carry this idea further. Given L/K a G-Galois extension as above with valuation ring extension B/A, for each $i \in \mathbb{Z}$, let

$$G_i = \{g \in G \,|\, \forall b \in B,\ v(g(b) - b) \geq i + 1\}. \tag{1}$$

Problem 5. Show that $G_{-1} = G$, that G_0 is as it was defined above, and that G_i is a normal subgroup of G for all i.

Problem 6. Show that, in (1), one need only check some $b \in B$ such that $B = A[b]$.

The filtration G_i is called the *higher ramification filtration for the lower numbering.* It is quite important because of the following theorem.

Theorem. If L/K is a G-extension as above, and L has maximal ideal m_L, then the *different* ideal $\delta_{L/K}$ of L/K is m_L^u, where

$$u = \sum_{i=0}^{\infty} |G_i| - 1.$$

The norm of the different is the *discriminant* of the extension. We will not say much about this. Note that there is also a *higher ramification filtration for the upper numbering.* There will be no problems about this, but we can discuss it if there is demand.

Problem 7. Show that, in the situation above, G_0/G_1 injects into $\ell^\times$ as a group, where ℓ is the residue field of L. *Hint:* Show that the map $\theta_0 : G_0/G_1 \to \ell^\times$ given by $\overline{g} \mapsto \overline{g(\pi_L)/\pi_L}$ does the trick. Here $\overline{g}$ is the image of $g \in G_0$ in G_0/G_1.

Problem 8. In the situation above, for $i \geq 1$, show that G_i/G_{i+1} injects into ℓ^+ as a group. *Hint:* Show that the map $\theta_i : G_i/G_{i+1} \to \ell^+$ given by $\overline{g} \mapsto \overline{(g(\pi_L) - \pi_L)/\pi_L^{i+1}}$ does the trick. Here $\overline{g}$ is the image of $g \in G_i$ in G_i/G_{i+1}.

Problem 9. Show that if the residue field k of K has characteristic 0, then G is cyclic. Show that if k has characteristic p, then $G_0 \cong G_1 \rtimes C$, where C is cyclic. Show that G_1 is a p-group and $p \nmid |C|$.

The group G_1 is often called the *wild inertia group* of L/K. If G_1 is trivial, the extension L/K is said to be *tamely ramified.*

Problem 10. Let K be a complete DVR with algebraically closed residue field of characteristic p. If L/K is a separable extension with prime-to-p order (not assumed to be Galois), then show that L/K is, in fact, Galois.

Problem 11. What is the absolute Galois group of $\mathbb{C}((t))$?

Problem 12. For an arbitrary $a \in \mathbb{Q}_2$, compute the higher ramification groups and different of $\mathbb{Q}_2(\sqrt{a})$. If this is too hard, assume $a \in \mathbb{Z}$.

Problem 13. Let ζ_p be a pth root of unity. Compute the higher ramification groups and different of $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$. What about $\mathbb{Q}_p(\zeta_p, \sqrt[p]{p})/\mathbb{Q}_p$?

Problem 14. Prove the following version of *Abhyankar's Lemma*: Let K be a complete DVR with perfect residue field. Let L/K and M/K be Galois extensions of K with $e_{L/K} | e_{M/K}$, and $p \nmid e_{M/K}$. Then LM/M is unramified.

Problem Set 3: Artin/Swan Representations/Conductors

Recall that if G is a finite group and $\rho : G \to GL_n(\mathbb{C})$ is a representation, then the *character* of ρ (written χ_ρ) is a map $G \to \mathbb{C}$ given by $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$.

Problem 1. Show that $\chi_\rho(g) = \overline{\chi_\rho(g^{-1})}$.

Note that a character is a class function, that is, it is constant on conjugacy classes in G. One can ask, given a class function on G, when is it in fact the character of a representation? One knows that any character is the character of at most one representation.

Let G be a finite group, and let L/K be a G-Galois extension of complete DVRs. Assume that the residue field k of K is perfect. The *Artin class function* a_G is defined by

$$a_G(g) = \begin{cases} -f_{L/K} \max(\{i \,|\, g \in G_{i-1}\}) & g \neq id \\ -\sum_{\substack{g \in G \\ g \neq id}} a_G(g) & g = id. \end{cases}$$

The *Swan class function* sw_G is defined by $sw_G(g) = a_G(g) + 1$ for $g \neq id$ and $sw_G(id) = a_G(id) + 1 - |G|$.

A major theorem is that the Artin class function and the Swan class function are actually each the character of a representation! Thus, they are usually referred to as the Artin character and the Swan Character.

Problem 2. For the extensions in Problems 12 and 13 of Problem Set 2, calculate the Artin and Swan characters, and find the corresponding representations.

Let $\rho : G \to GL(V)$ be a $\mathbb{C}$-representation of G, and let χ_ρ be its character. Then the *Artin conductor* of χ_ρ, is given by

$$\langle a_G, \chi_\rho \rangle := \frac{1}{|G|} \sum_{g \in G} a_G(g)\overline{\chi_\rho(g)}.$$

The *Swan conductor* of χ_ρ is given by

$$\langle sw_G, \chi_\rho \rangle := \frac{1}{|G|} \sum_{g \in G} sw_G(g)\overline{\chi_\rho(g)}.$$

One can show that the Artin conductor of χ_ρ is in fact equal to

$$\sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|}(\dim V - \dim V^{G_i}),$$

where V^{G_i} is the subspace of V fixed pointwise by G_i.

Problem 3. Given what we have said so far, show that the Swan conductor of χ_ρ is equal to

$$\sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|}(\dim V - \dim V^{G_i}).$$

Problem 4. If $G = \text{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$, let $\rho = \chi_\rho : G \to \mathbb{C}^\times$ be the cyclotomic character. Calculate its Artin and Swan conductors by both methods.

Problem 5. If $G = \text{Gal}(\mathbb{Q}_p(\zeta_p, \sqrt[p]{p})/\mathbb{Q}_p)$, let $\rho : G \to GL_{p-1}(\mathbb{C})$ be the unique irreducible faithful representation, up to isomorphism. Calculate the Artin and Swan conductors of its character χ_ρ.

Problem Set 4: Ramification in Characteristic p

Throughout this problem set, k is an algebraically closed field of characteristic p.

Let K be a field of characteristic p. The *Artin-Schreier theory* says that all $\mathbb{Z}/p$-extensions of K are of the form $K[y]/(y^p - y - a)$, where $a \in K$. The Galois action of a generator of $\mathbb{Z}/p$ is given by $y \mapsto y + 1$.

Problem 1. Fix an algebraic closure $\overline{K}$ of K. Show that if $a' = a + b^p - b$ for some $b \in K$, then $K[y]/(y^p - y - a) = K[y]/(y^p - y - a')$ as subextensions of $\overline{K}/K$.

Problem 2. Let $K = k((t))$. Fix an algebraic closure $\overline{K}$ of K. Show that every $\mathbb{Z}/p$-extension of K inside $\overline{K}$ can be given by $K[y]/(y^p - y - a)$, where $a = \sum_{i=1}^{n} a_i t^{-i}$, with $a_i \in k$, and $a_i = 0$ when $p \mid i$. *Hint:* First show that every power series in t is of the form $b^p - b$.

Problem 3. (From [BW]) Fix $h \in \mathbb{N} \backslash p\mathbb{N}$. Let k be algebraically closed of characteristic p, let $A = k[[t]]$, let $K = k((t))$, and let $L = K[y]/(y^p - y - t^{-h})$. Let B be the integral closure of A in L.

(a) Find $z \in B$ such that $B = k[[z]]$.

(b) If σ is a generator of $\mathrm{Gal}(L/K)$, write down a power series for $\sigma(z)$.

(c) Determine the higher ramification filtration of L/K.

One major difference between characteristic p and characteristic zero is that, in characteristic p, there exist nontrivial étale covers of the affine line (if you don't know what étale means, then for our purposes such a cover is a finite map $Y \to \mathbb{A}^1_k$ of smooth k-varieties such that the cardinality of each fiber is the same. See also Problem Set 10). If $\mathbb{A}^1_k$ is Spec $k[x]$, then the function field of $\mathbb{A}^1_k$ is $k(x)$. If $Y \to \mathbb{A}^1_k$ is a finite morphism, and Y is normal, then $Y =$ Spec B, where B is the integral closure of $k[x]$ in a field extension of $k(x)$. The map $f : Y \to \mathbb{A}^1_k$ is called a *Galois cover* (with group G) if this field extension is Galois (with group G). The G-action on B yields a G-action on Y that preserves each fiber of f.

Problem 4. Let $L/k(x)$ be the field extension given by $y^p - y = x$. If $Y \to \mathbb{A}^1$ is the corresponding cover, show that it is étale and that Y is isomorphic to $\mathbb{A}^1$.

Recall the *Hurwitz formula*. This says that if $f : Y \to X$ is a finite, generically separable map of smooth projective k-curves of degree d, then

$$2g(Y) - 2 = d(2g(X) - 2) + |R|,$$

where $g(X)$ and $g(Y)$ are the respective genera of X and Y, and R is the *ramification divisor* on Y. The ramification divisor is given as follows: If $y \in Y$ is a closed point, and the different of $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,f(y)}$ is m_y^i, where m_y is the maximal ideal of $\hat{\mathcal{O}}_{Y,y}$, then the coefficient of y in R is i. The inertial degree of $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,f(y)}$ is called the *ramification index* at y.

Problem 5. Show that if the ramification index e_y at y is prime to the characteristic of k, then the coefficient of y in R is $e_y - 1$. (Use the theorem from Problem Set 2).

If k is $Y \to \mathbb{A}^1_k$ is an étale cover induced from a field extension $L/k(x)$, then $\overline{Y} \to \mathbb{P}^1$ is given by taking the normalization of $\mathbb{P}^1$ in $L/k(x)$. It is étale, except perhaps at ∞.

Problem 6. Let $h \in \mathbb{N} \backslash p\mathbb{N}$.

(a) Show that if $L = k(x)[y]/(y^p - y - x^h)$, then $L/k(x)$ is Galois with group $\mathbb{Z}/p$, and gives an étale Galois cover $Y \to \mathbb{A}^1_k$.

(b) Let $\overline{Y} \to \mathbb{P}^1_k$ be the corresponding projective cover. What is the genus of $\overline{Y}$?

Problem 7. Give the weakest condition you can on a finite group G that would prohibit it from being the group of a Galois étale cover of $\mathbb{A}^1_k$.

Problem Set 5: Adeles and Heights

Let K be a number field, and $\mathcal{O}_K$ its ring of integers. For each prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, the localization of $\mathcal{O}_K$ at $\mathfrak{p}$ is a DVR, inducing a valuation $v_\mathfrak{p}$ on K, and thus an absolute value by $|x|_\mathfrak{p} = (N\mathfrak{p})^{-v(x)}$, where $N\mathfrak{p}$ is the order of $K/\mathfrak{p}$. Each embedding of K into $\mathbb{R}$ or pair of complex conjugate embeddings into $\mathbb{C}$ also gives rise to an absolute value on K, by restricting the standard absolute value. The collection of all these absolute values is called the set of *places* of K (really, places are equivalence classes of absolute values, but this won't matter for us). The absolute values coming from embeddings into $\mathbb{R}$ or $\mathbb{C}$ are called *archimedean* places, and the others are called *non-archimedean*. Write V_K for the set of places of K. If $v \in V$ is non-archimedean, corresponding to a prime $\mathfrak{p}$, write $\mathcal{O}_v$ for the completion of the localization of $\mathcal{O}_K$ at v, and write $K_v = \mathrm{Frac}(\mathcal{O}_v)$. If $v \in V_K$ is real (resp. complex), write $\mathcal{O}_v = K_v = \mathbb{R}$ (resp. $\mathcal{O}_v = K_v = \mathbb{C}$).

Recall the definition of the *adeles* $\mathbb{A}_K$ of K from [CS]:

$$\mathbb{A}_K = \left\{ (x_v) \in \prod_{v \in V_K} K_v \; : \; x_v \in \mathcal{O}_v \text{ for all but finitely many } v \in V_K \right\}.$$

The adeles are clearly a ring, and have a topology given by taking a basis of neighborhoods of zero to be of the form

$$\prod_{v \in V_K} U_v,$$

where U_v is open and $U_v = \mathcal{O}_v$ for all but finitely many $v \in V_K$. The adeles are a topological ring under this topology.

For each complex place v, fix one of the complex conjugate embeddings $K \to K_v$ corresponding to v. Then the embeddings $K \to K_v$ for each v give rise to an embedding $K \to \mathbb{A}_K$.

Problem 1. (From [CS]) Show that K embeds *discretely* into $\mathbb{A}_K$. *Hint:* Use the fact that $\mathcal{O}_K$ embeds discretely into the product of the K_v for v arechimedean.

Problem 2. Show that, as groups, we have $\mathbb{A}_\mathbb{Q}/\mathbb{Q} \cong S^1 \times \hat{\mathbb{Z}}$. Are they isomorphic as topological groups?

In fact, $\mathbb{A}_K/K$ is always compact, but this is a little more involved.

Problem 3. Show that $\mathbb{A}_K$ is not connected.

For each $v \in V_K$, let $|\cdot|_v$ be the absolute value corresponding to v.

Problem 4. (Product formula) Let $x \in K^\times$. Show that $\prod_{v \in V_K} |x|_v^{n_v} = 1$, where $n_v = 2$ if v is complex, and $n_v = 1$ otherwise. *Hint:* First assume $K = \mathbb{Q}$. Then, for arbitrary K, show that the product formula in $\mathbb{Q}$ for the norm $N(x)$ gives the product formula for x.

If $x \in K^\times$ then the *height* $H_k(x)$ is given by $\prod_{v \in V_K} \max(1, |x|_v)$.

Problem 5. (From [CS]) If $x = r/s$, with r and s relatively prime integers and $s \neq 0$, show that $H_\mathbb{Q}(x) = \max(|r|, |s|)$.

Problem Set 6: Central Simple Algebras

If K is a field, then a *central simple algebra (csa)* over K is a finite associative (not necessarily commutative) unital K-algebra with no nontrivial two-sided ideals, whose center is K.

Problem 1. Show that, for any n, $\mathrm{Mat}_n(K)$ is a csa over K.

Problem 2. Show that if D is a division algebra with center K, then D is a csa over K.

In fact, a theorem of Wedderburn states that *all* csa's over K are of the form $\mathrm{Mat}_n(D)$, where D is a division algebra with center K. Also, if $\mathrm{Mat}_n(D) \cong \mathrm{Mat}_{n'}(D')$, then $n = n'$ and $D \cong D'$.

If K is a field, and $a, b \in K^\times$, then we define the *quaternion algebra* $A(a, b, K)$ to be the central K-algebra with underlying vector space $K \oplus Ki \oplus Kj \oplus Kij$ subject to the multiplication law $i^2 = a$, $j^2 = b$, and $ij = -ji$. By dimension count reasons, it is clear that $A(a, b, K)$ is either a division algebra or isomorphic to $M_2(K)$.

Problem 3. Show that $A(a, b, K)$ is a central simple algebra.

Problem 4. Let $K = \mathbb{Q}_p$. For which choices of A and B in $K \backslash K^2$ is $A(a, b, K)$ a division algebra? What about for $K = \mathbb{R}$?

Problem 5. Let $a, b \in \mathbb{Q}^\times$. Prove that $A(a, b, \mathbb{R})$ is a division algebra iff $A(a, b, \mathbb{Q}_p)$ is a division algebra for an *odd* number of p. *Hint:* Use quadratic reciprocity. This is a special case of the Brauer-Hasse-Noether Theorem.

Problem 6. Let D be a division algebra with center K, and let D^{op} be the opposite ring (same underlying additive group, but ab in D^{op} is equal to ba in D). Show that $D \otimes_K D^{op} \cong \mathrm{Mat}_n(K)$, where $n = [D : K]$. *Hint:* Consider the action of $D \otimes_K D^{op}$ on A given by $(a \otimes b)(c) = acb$. You may use without proof the fact that a tensor product of two csa's over K is a csa over K.

Problem 7. For any quaternion algebra $A(a, b, K)$ over any field, show that $A \cong A^{op}$.

Recall that two csa's A and B over K are *Brauer equivalent* if $A = \mathrm{Mat}_n(D)$ and $B = \mathrm{Mat}_{n'}(D)$ for the same division algebra D. In light of Problem 6, the Brauer equivalence classes of csa's over K form an abelian group under tensor product, with the class of K as the identity element. This group is called the Brauer group of K, and written $Br(K)$. Note that each Brauer class has exactly one element that is a division algebra.

Problem 8. Show that the Brauer group of an algebraically closed field is trivial.

Problem 9. Determine the Brauer group of any field $\mathbb{F}_q$.

Problem Set 7: Some Basic Scheme Theory

Of course, we cannot give a full introduction to scheme theory. We will just touch on a few points that will be relevant to the lectures. Let A be a ring. The set of prime ideals of A is denoted Spec A, and the topology on Spec A is given by taking closed sets to be those of the form $V(f) := \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p}\}$ (it is not hard to verify that this is a topology).

Problem 1. Show that the closed points of Spec A are the maximal ideals.

Problem 2. Show that if A is a domain, then there exists a *generic point* $\eta \in \text{Spec } A$, i.e., the closure of η is Spec A.

Problem 3. If A is a DVR, describe the topological space Spec A completely.

By the Nullstellensatz, if A is a finitely generated, reduced, algebra over an algebraically closed field, the subspace of Spec A consisting of the maximal ideals is homeomorphic to the variety with coordinate ring A in the Zariski topology. You can verify a few examples if you wish. The standard topological results carry over (i.e., Spec A is irreducible iff A is a domain, etc.). The *dimension* of Spec A is the Krull dimension of A, see [HH] for more details. The affine line over a ring R is $\mathbb{A}^1_R = \text{Spec } R[x]$. See [HH] for how $\mathbb{P}^1_R$ is formed by gluing two copies of $\mathbb{A}^1_R$. Note that an affine scheme is the topological space Spec A, along with a sheaf of functions. We will not give details now, except to say that the ring of regular functions on Spec A is A. The value of a function a on Spec A at a point $\mathfrak{p}$ is the element of $A/\mathfrak{p}$ given by the reduction of a. For $A = \mathbb{C}[x]$, for instance, it is a good idea to convince yourself that this is essentially the same as the standard way of thinking about functions. The ring of *rational functions* on Spec A is the total ring of fractions of A. If A is a domain, this is Frac(A). If A is a domain, the rings of rational functions on $\mathbb{A}^1_A$ and $\mathbb{P}^1_A$ are both Frac$(A)(x)$.

Problem 4. (From [HH]) Let k be a field, and let T be either $k[[t]]$ or $\mathbb{Z}_p$. Describe the topological spaces $\mathbb{A}^1_T$ and $\mathbb{P}^1_T$ in as much detail as you can. What are the closed subsets? To make things easier at first, you might assume k algebraically closed. What are the dimensions of these spaces?

Problem 5. For $T = k[[t]]$ (resp. $\mathbb{Z}_p$), show that every closed point in $\mathbb{P}^1_T$ lies in the zero locus of the ideal (t) (resp. (p)) (note that both t and p are global functions on $\mathbb{P}^1_T$). If X is the zero locus of (t) (resp. (p)), show that X is isomorphic to $\mathbb{P}^1_k$ (resp. $\mathbb{P}^1_{\mathbb{F}_p}$) and the complement of X in $\mathbb{P}^1_T$ is isomorphic to $\mathbb{P}^1_{k((t))}$ (resp. $\mathbb{P}^1_{\mathbb{Q}_p}$). If you don't know what isomorphism of schemes is, then just show homeomorphism.

Let $X \in \text{Spec } A$, and let $x \in X$ be a point corresponding to a prime ideal $\mathfrak{p}$. Then the local ring of X at x, denoted $\mathcal{O}_{X,x}$, is the ring $A_{\mathfrak{p}}$. If X is a general scheme, and $x \in X$, then there is an open subset Spec $A \subseteq X$ containing x. Then x corresponds to a prime ideal $\mathfrak{p}$ of A, and $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$. The completion of $\mathcal{O}_{X,x}$ for the $\mathfrak{p}$-adic topology is called the complete local ring of X at x, and is denoted $\mathcal{O}_{X,x}$.

Problem 6. Let $T = k[[t]]$, with k algebraically closed, and let $X = \text{Spec } T[x] = \mathbb{A}^1_T$. For each point x of X, compute $\mathcal{O}_{X,x}$ as explicitly as possible.

If some of the questions do not make much sense to people who have not yet seen scheme theory, I am happy to discuss schemes more thoroughly.

Problem Set 8: Fuchsian Groups and the Hyperbolic Plane

As in [CS], we write $\mathbf{H} := \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$ for the hyperbolic plane. The metric on $\mathbf{H}$ is $|dz|/\mathrm{Im}\ z$, or in other words, the area element is $(dx^2 + dy^2)/y^2$. The group $PSL_2(\mathbb{R})$ acts on $\mathbf{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

It is the isometry group of $\mathbf{H}$.

Problem 1. (From [CS]) Let $\Gamma \leq PSL_2(\mathbb{R})$ be a subgroup. Prove that $\Gamma\backslash\mathbf{H}$ is Hausdorff iff Γ is discrete in $PSL_2(\mathbb{R})$ (here $PSL_2(\mathbb{R})$ has the quotient topology from $SL_2(\mathbb{R})$, which has the subspace topology from $\mathbb{R}^4$).

Problem 2. (From [CS]) Show that any cocompact Fuchsian group Γ (i.e., $\Gamma\backslash\mathbf{H}$ is compact) has a torsion-free subgroup of finite index.

One can also consider the unit disk model $\mathbf{D}$ of the hyperbolic plane, where $\mathbf{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. The map $f(z) = \frac{iz+1}{z+i}$ gives an isomorphism $\mathbf{H} \to \mathbf{D}$, and $\mathbf{D}$ inherits the metric of $\mathbf{H}$ via f.

Problem 3. (From [CS]) Write the metric on $\mathbf{D}$ in terms of dz.

Problem 4. (From [CS]) Show that the isometry group of $\mathbf{D}$ is the projective unitary group $PU(1,1) := \{A \in PSL_2(\mathbb{C}) \,|\, {}^t\overline{A}hA = h\}$, where h is the Hermitian form on $\mathbb{C}^2$ with matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Here, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A$, then $\gamma z = \frac{az+b}{cz+d}$.

Let Γ be a Fuchsian group. Then a *fundamental domain for* Γ is a subset $\Delta \subseteq \mathbf{H}$ such that, for each $z \in \mathbf{H}$, there exists a unique $\gamma \in \Gamma$ such that $\gamma z \in \Delta$.

Problem 5. Let S be the set $\{z = x + iy \in \mathbf{H} \,|\, |z| = 1 \wedge x > 0\}$. Show that the set $\{z = x + iy \in \mathbf{H} \,|\, -\frac{1}{2} \leq x < \frac{1}{2} \wedge |z| \geq 1\}\backslash S$ is a fundamental domain for $SL_2(\mathbb{Z})$.

Problem 6. The group $\Gamma_0(N)$ is the set of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $c \equiv 0 \pmod{N}$. Find a fundamental domain for $\Gamma_0(2)$.

Problem Set 9: Basics on Stable Curves and p-adic Geometry

Let R be a DVR with fraction field K, and let X be a smooth K-curve. Then a *smooth model* for X over R is a smooth map $X_R \to \operatorname{Spec} R$ of dimension 1 such that the generic fiber is X. For instance, $\mathbb{P}^1_R$ is a smooth model for $\mathbb{P}^1_K$. If a curve over K has a smooth model, it is said to have *good reduction*.

Problem 1. Let R be a DVR, let $\mathbb{P}^1_R$ be given by the gluing of Spec $R[x]$ and Spec $R[x^{-1}]$, and let $z \in \mathbb{P}^1_R$ be the closed point on the special fiber corresponding to $x = a$, for some a in the residue field k of R. Describe the set of points of the generic fiber that *specialize* to p, i.e., whose closures contain the point z. What about when z corresponds to the point "$x = \infty$" (i.e., $x^{-1} = 0$)?

Problem 2. (From [HH]) Let R be any ring with a nonarchimedean absolute value (see [HH]). Consider the set $R\{x\} \subseteq R[[x]]$ consisting of power series whose coefficients approach 0 in the topology of R.

(a) Show that $R\{x\}$ is a ring.

(b) Show that $f(x) \in R[[x]]$ lies in $R\{x\}$ iff it converges for every $|x| \leq 0$.

(c) Give a condition on the coefficients of $f(x) \in R[[x]]$ that is equivalent to $f(x)$ converging for all $|x| \geq 0$.

(d) If $R = k[[t]]$, show that $R\{x\} \cong k[x][[t]]$.

(e) Let $R = k[[t]]$ in the situation of Problem 1. Consider the ring R_U of rational functions on $\mathbb{P}^1_R$ that are regular on $U = \operatorname{Spec} k[x] \subseteq \operatorname{Spec} R[x] \subseteq \mathbb{P}^1_R$. Show that the completion of this ring with respect to the t-adic topology is equal to the ring in part (b) (equivalently, part (d)–this is the ring $\hat{R}_U$ in [HH]). State and prove something analogous when $R = \mathbb{Z}_p$.

In light of part (b), we often think of $R\{x\}$ as the ring of R-valued analytic functions on the closed unit disk. Note that, in the situation of Problem 1, the set of points that specialize to $U = \operatorname{Spec} k[x]$ form (in some sense), a closed unit disk. Thus the set of analytic functions on the closed unit disk is equal to the t-adic completion on R_U. This is a simple case of the equivalence between "rigid geometry" and "formal geometry."

A *semi-stable curve* over a field k is a curve whose only singularities are ordinary double points (i.e., the complete local ring of the singularity is of the form $k[[x,y]]/(xy)$, as if the singularity were the one at the intersection of the two axes in $\mathbb{A}^2_k$). If R is a DVR with fraction field K, and X is a smooth K-curve, then a *semi-stable* model for X is flat map $X \to \operatorname{Spec} R$ with semi-stable special fiber. It is a famous theorem that every smooth K-curve has a semi-stable model after base changing to a finite extension of K.

We will only discuss semi-stable models of $\mathbb{P}^1_K$, given by the standard gluing of Spec $K[x]$ and Spec $K[x^{-1}]$. For simplicity, we will assume the residue field k of K is algebraically closed.

Let $\mathbb{P}^1_R$ be the smooth model form $\mathbb{P}^1_K$ that is the standard gluing of $R[x]$ and $R[x^{-1}]$.

Problem 3. Let π be a uniformizer of R. Consider the blow-up V of $\mathbb{P}^1_R$ corresponding to the ideal $(\pi, x) \subseteq R[x]$.

(a) Show that the special fiber of V is isomorphic to two projective lines meeting at a point, and the generic fiber of V is isomorphic to $\mathbb{P}^1_K$.

(b) Determine which K-points of the generic fiber specialize to which of the two components. Do any K-points specialize to the singular point? Do any points of the generic fiber specialize to the singular point?

(c) Instead, suppose we blew up the ideal $(\pi^2, x) \in R[x]$. Now do any K-points specialize to the singular point?

(d) Suppose we wanted to make a further blowup in part (c) such that the K-points corresponding to $(x - \pi^2)$ and $(x - \pi^2 - \pi^3)$ would specialize to different points. What ideal would we blow up? Would we still have a semi-stable model of $\mathbb{P}^1_K$? Draw a diagram of the special fiber and mark the specializations of (x), $(x - \pi)$, $(x - \pi^2)$, $(x - \pi^2 - \pi^3)$, and $(\frac{1}{x})$.

Problem 4. Notation as above. Let $\{z_1, \ldots, z_n\}$ be a subset of $K \cup \{\infty\}$. Show that there exists a semi-stable model of $\mathbb{P}^1_K$ such that the points $(x - z_i)$ all specialize to different points on the special fiber (here we take $(x - \infty)$ to be the ideal $(\frac{1}{x})$).

Problem 5. (Stable reduction theorem for the marked projective line). In the situation above, if $n \geq 3$, show that the semi-stable model can be chosen such that each irreducible component of the special fiber has at least three marked points (meaning singular points or specializations of an $(x - z_i)$). Show that no irreducible component of the special fiber of such a model can be contracted while preserving this property.

In fact, the model above is called the *stable model* of the marked curve $\mathbb{P}^1_K$ marked by the points $(x - z_i)$. It is unique up to unique isomorphism (for an appropriately defined notion of isomorphism...we can discuss if you want details).

Problem Set 10: Étale Morphisms and the Étale Topology

Problem 1. Let X be an irreducible variety over an algebraically closed field k. Show that if Y is a connected k-variety with a morphism $f : Y \to X$ that is a topological cover (using the Zariski topology), then f is an isomorphism.

The above problem shows that covering spaces for the Zariski topology are often not so interesting! However, the theory of *étale morphisms* is much richer, and plays the role of covering spaces (more specifically, local homeomorphisms) for algebraic geometry. We will not give much exposition here. A good reference is James Milne's notes on étale cohomology, which can be found at

http://www.jmilne.org/math/CourseNotes/lec.html.

We will, however, give a definition of an étale morphism:

Definition: A local morphism $f : A \to B$ of local rings is *unramified* if $f(m_A)B = m_B$, and if $A/m_A \to B/m_B$ is a finite, separable extension. A morphism $f : Y \to X$ of schemes is *étale* if it is flat, of finite type, and for all points $y \in Y$, the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is unramified.

For those unfamiliar with schemes, the definition still works if "schemes" is replaced with "varieties over an algebraically closed field." One sees easily that isomorphisms and open immersions are étale, and that compositions of étale morphisms are étale.

Problem 2. Let X be a K-scheme, for some field K. Let L/K be a finite, separable extension. Show that $X \times_K L \to X$ is étale.

Problem 3. Let $f : Y \to X$ be a finite, étale morphism of smooth $\mathbb{C}$-varieties. Show that $f(\mathbb{C}) : Y(\mathbb{C}) \to X(\mathbb{C})$ is a finite cover (for the *complex* topology). *Hint:* It is enough to show that $f(\mathbb{C})$ is a local homeomorphism with constant fiber cardinality. Now use the inverse function theorem, along with the the fact that the tangent space of a variety at a point p is the dual of m_p/m_p^2, where m_p is the ideal of functions vanishing at that point. *Note:* One need not assume smoothness, but then proof is harder.

Problem 4. Show that if $X \to \text{Spec}\ (\mathbb{Z})$ is a finite étale morphism, then $X \cong \coprod_{i=1}^n \text{Spec}\ \mathbb{Z}$.

Problem 5. Give an example of a map $f : Y \to X$ of $\mathbb{C}$-varieties that is not étale, but such that the map $f(\mathbb{C}) : Y(\mathbb{C}) \to X(\mathbb{C})$ on $\mathbb{C}$-points *is* a cover for the complex topology.

Since topological covers in the Zariski topology are so rare, one instead uses étale morphisms to define the fundamental group. Let X be a scheme. A finite morphism $f : Y \to X$ of connected schemes is *Galois* if $|\operatorname{Aut}(Y/X)| = \deg(f)$. If $f : Y \to X$ and $g : Z \to X$ are finite Galois étale morphisms of schemes, then there is a finite Galois étale morphism $h : W \to X$ dominating f and g (one can take W to be a connected component of $Y \times_X Z$). Thus the finite Galois étale morphisms $X_i \to X$ form an inverse system. So do the groups $\operatorname{Gal}(X_i/X)$, and we define the fundamental group $\pi_1(X)$ to be $\varprojlim_i \operatorname{Gal}(X_i/X)$, where X_i ranges over all finite covers. By construction, it is a finite group.

Problem 6. Show that if k is algebraically closed of characteristic zero, then $\pi_1(\mathbb{P}^1_k)$ and $\pi_1(\mathbb{A}^1_k)$ are both trivial.

Problem 7. What is $\pi_1(\mathbb{A}^1_k \backslash \{0\})$, with k as above?

Problem 8. What is $\pi_1(\text{Spec } \mathbb{F}_p)$?

Recall that an *open covering* of a topological space X is a collection of open immersions $u_i : X_i \to X$ such that $\bigcup_i u_i(X_i) = X$. Note that open coverings satisfy the following three axioms:

(i) $id : X \to X$ is an open covering.

(ii) If the collection $u_i : X_i \to X$ is an open covering, and for each i, there is an open covering $u_{ij} : X_{ij} \to X_i$, then the collection $u_i \circ u_{ij} : X_{ij} \to X$ is an open covering.

(iii) If the collection $u_i : X_i \to X$ is an open covering, and $Y \subseteq X$ is any subset, then the collection $u_i|_Y : X_i \cap Y \to Y$ is an open covering.

Now, let X be a scheme. We call a collection of morphisms $f_i : X_i \to X$ an *étale covering* if the f_i are all étale, and the union $\bigcup_i (f_i(X_i)) = X$. Clearly, a open covering for the Zariski topology is also an étale covering.

Problem 9. Show that étale coverings satisfy the same axioms as above, where for axiom (iii), we allow any morphism $f : Y \to X$, replacing $X_i \cap Y$ by $X_i \times_X Y$.

One says that étale coverings of X yield a Grothendieck topology, called the *étale topology.* Of course, this is not really a topology, but it will allow us to define sheaves on the scheme X. Let $\mathcal{C}$ be the category of étale schemes over X (that is, objects are étale morphisms $f : Y \to X$, and a morphism from $(f : Y \to X) \to (g : Z \to X)$ is a map $h : Y \to Z$ such that $g \circ h = f$—it turns out that h will be étale). Then a contravariant functor

$$\mathcal{F} : \mathcal{C} \to \text{Sets}$$

is called a sheaf (for the étale topology) if it satisfies the definition given on the first page of [MS]. It turns out that instead of checking every étale covering, one need only check Zariski open coverings, and coverings consisting of a single étale morphism $V \to U$ when both V and U are affine. Note that this means that if $V = \text{Spec } B$ and $U = \text{Spec } A$, then B is faithfully flat over A.

Problem 10. Show that the functor taking $f : Y \to X$ to $\Gamma(Y, \mathcal{O}_y)$ is a sheaf for the étale topology. *Hint:* It is a Zariski sheaf essentially by definition. Show that the condition in [MS] for single étale morphisms of affines is equivalent to showing that if B is a faithfully flat A-algebra, then the sequence

$$0 \to A \to B \to B \times_A B$$

is exact, where the last map sends b to $1 \otimes b - b \otimes 1$. To do this, first show that it is true when $A \to B$ admits a section, and then reduce to this case.