p-adic cohomology from theory to practice
Lecture 1: Algebraic de Rham cohomology

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Differentials

Let $X = \operatorname{Spec} A$ be an affine variety over a field K of characteristic zero. Let $\Omega^1_{A/K}$ denote the module of Kähler differentials; that is, $\Omega^1_{A/K}$ is the A-module generated by symbols da for $a \in A$, modulo the relations da for $a \in K$, and $d(ab) - a\,db - b\,da$ for $a, b \in A$.

We assume hereafter that X/K is smooth, which forces $\Omega^1_{A/K}$ to be a locally free A-module. Put

$$\Omega^i_{A/K} = \wedge^i_A \Omega^1_{A/K};$$

that is, $\Omega^i_{A/K}$ is the free A-module generated by symbols $\omega_1 \wedge \cdots \wedge \omega_i$, modulo the relations

$$(a\omega_1 + a'\omega_1') \wedge \omega_2 \wedge \cdots \wedge \omega_i - a\omega_1 \wedge \omega_2 \cdots \wedge \omega_i - a'\omega_1' \wedge \omega_2 \cdots \wedge \omega_i$$

for $a, a' \in A$, and $\omega_1 \wedge \cdots \wedge \omega_i = 0$ whenever two of the factors are equal.

The de Rham complex

The map $d : A \to \Omega^1_{A/K}$ sending a to da induces maps $d : \Omega^i_{A/K} \to \Omega^{i+1}_{A/K}$. Moreover, the composition $d \circ d$ is always zero.

We thus have a complex $\Omega^{\cdot}_{A/K}$, called the *de Rham complex* of A (or X). The cohomology of this complex is called the *(algebraic) de Rham cohomology* of A, or X, denoted $H^i_{\mathrm{dR}}(X)$.

Note that $H^i_{\mathrm{dR}}(X)$ vanishes above the dimension of X, because the complex itself vanishes there.

Sheaf cohomology for a single sheaf

Let $\{U_i\}_{i \in I}$ be a finite cover of X by open affine subschemes. For $j = 0, 1, \dots,$ let I_j be the set of $(j+1)$-element subsets of I. For $J \in I_j$, let X_J be the the intersection of the U_i for $i \in J$.

For a quasicoherent sheaf $\mathscr{F}$ on X, the corresponding Čech complex $\check{C}^{\cdot}(X, \mathscr{F})$ has j-th term $\prod_{J \in I_j} \Gamma(X_J, \mathscr{F})$, and

$$d((s_J)_{J \in I_j}) = \left(\sum_{i=0}^{j+1} (-1)^i s_{J - \{j_i\}} \right)_{J \in I_{j+1}},$$

where the elements of $J \in I_{j+1}$ are labeled $j_0, \dots, j_{j+1}$ in increasing order. Its cohomology computes the sheaf cohomology $H^i(X, \mathscr{F})$.

Hypercohomology: sheaf cohomology for a complex

Create the Čech complex for each term of the de Rham complex for X. You end up with a *double complex*:

$$\begin{array}{ccccccc}
\check{C}^0(X,\mathscr{O}_X) & \longrightarrow & \check{C}^1(X,\mathscr{O}_X) & \longrightarrow & \check{C}^2(X,\mathscr{O}_X) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\check{C}^0(X,\Omega^1_{X/K}) & \longrightarrow & \check{C}^1(X,\Omega^1_{X/K}) & \longrightarrow & \check{C}^2(X,\Omega^1_{X/K}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\check{C}^0(X,\Omega^2_{X/K}) & \longrightarrow & \check{C}^1(X,\Omega^2_{X/K}) & \longrightarrow & \check{C}^2(X,\Omega^2_{X/K}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Hypercohomology (continued)

Let d_X and $\check{d}$ denote the vertical and horizontal maps, respectively. To take cohomology here, you take cohomology of the total complex in which

$$C^i = \oplus_{j+k=i} \check{C}^j(X, \Omega^k_{X/K}),$$

with the differentials

$$d((\omega_{j,k})_{j+k=i}) = (d_X(\omega_{j,k-1}) + (-1)^j \check{d}(\omega_{j-1,k}))_{j+k=i+1}.$$

Comparing analytic and algebraic cohomology

For $K = \mathbb{C}$, there is a natural map

$$H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}}(X^{\mathrm{an}})$$

because we can use the same recipe to compute holomorphic de Rham cohomology.

For X projective (or even proper), GAGA implies that this map is an isomorphism. Using this, Grothendieck showed that $H^i_{\mathrm{dR}}(X)$ is finite dimensional for arbitrary X.

For K a complete nonarchimedean field, analogous statements are true; this is what makes p-adic cohomology easy to compute.

Log-differentials

A *smooth (proper) pair* over some base S is a pair (X,Z) in which X is a smooth (proper) scheme over S and Z is a relative (to S) strict normal crossings divisor.

Over a field: each component of Z is smooth, and the components of Z always meet transversely.

In general, étale locally over S, X should look like an affine space and Z should look like an intersection of coordinate hyperplanes.

Log-differentials (contd.)

Let (X,Z) be a smooth pair over a field K. Put $U = X - Z$ and let $j : U \hookrightarrow X$ be the implied open immersion.

The *sheaf of logarithmic differentials* on X, denoted $\Omega^1_{(X,Z)/K}$, is the subsheaf of $j_*\Omega^1_{U/K}$ generated by $\Omega^1_{X/K}$ and by sections of the form df/f, where f is a regular function on some open subset V of X which only vanishes along components of Z.

Again, we write $\Omega^i_{(X,Z)/K}$ for the i-th exterior power of $\Omega^1_{(X,Z)/K}$ over $\mathscr{O}_X$.

Logarithmic de Rham cohomology

The obvious map of complexes

$$\Omega^{\cdot}_{(X,Z)/K} \to j_* \Omega^{\cdot}_{U/K}$$

is a quasi-isomorphism, i.e., it induces isomorphisms on cohomology sheaves. Hence we obtain an isomorphism $\mathbb{H}^i(\Omega^{\cdot}_{(X,Z)/K}) \cong H^i_{\mathrm{dR}}(U)$.

If X is proper, logarithmic de Rham cohomology can be computed analytically, again by GAGA.