Stark-Heegner points
Arizona Winter School 2011

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Classical Heegner points

Let $E_{/\mathbb{Q}}$ be an elliptic curve and

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The modular parametrization is

$$\begin{array}{rcccl} \varphi : & X_0(N) & \longrightarrow & & E \\ & \infty & \mapsto & & 0 \\ & \tau & \mapsto & P_\tau := & 2\pi i \int_\infty^\tau f(z)dz \\ & & & = & \sum_{n\geq 1} \frac{a_n}{n} e^{2\pi i n \cdot \tau} \end{array}$$

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Put

$$\mathcal{O}_\tau = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : N \mid c, \gamma \cdot \left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right) = \lambda \left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)\} \subset \mathrm{M}_0(N) \subseteq M_2(\mathbb{Z}).$$

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$\mathcal{O}_\tau$ is an order in K in which all $p \mid N$ split or ramify, and

$$P_\tau \in E(H_{\mathcal{O}_\tau}),$$

where $\mathrm{Gal}(H_{\mathcal{O}_\tau}/K) \simeq \mathrm{Pic}(\mathcal{O}_\tau)$.

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Theorem (Gross-Zagier)

$L'(E/K, 1)/\Omega_E \doteq height(P_K)$ where $P_K = Tr_{H_{\mathcal{O}_\tau}/K}(P_\tau)$.

Corollary

$P_K \in E(K)$ has infinite order if and only if $L'(E/K, 1) \neq 0$.

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Theorem (+Kolyvagin)

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Corollary

If $r_{an}(E/\mathbb{Q}) \leq 1$, *BSD holds true for* $E/\mathbb{Q}$.

What about BSD when $\mathrm{ord}_{s=1} L(E/K, s)$ is odd but K fails to satisfy H.H.?

Heegner points on Shimura curves

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We *still* have $\varphi : X_0^{N^-}(N^+) \dashrightarrow E$, $[\tau] \mapsto P_\tau \in E(H_{\mathcal{O}_\tau})$. All works nicely thanks to Zhang.

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What can we say if any of these fails? How do we construct points on E over other fields?

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and

$$\begin{array}{ccccc} \mathrm{Pic}_0(X)(\mathbb{C}) & \overset{AJ}{\overset{\sim}{\longrightarrow}} & (H^{1,0})^\vee/H_1(X,\mathbb{Z}) & \simeq & \mathbb{C}^g/\Lambda \\ D & \mapsto & \int_D & \mapsto & (\int_D \omega_1, \ldots, \int_D \omega_g) \end{array}$$

$H^{1,0} := H^0(X_{\mathbb{C}}, \Omega^1)$.

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For non-split Shimura curves $X_0^{N^-}(N^+)$ there is no choice of a base point $\infty \in X(\mathbb{Q})$ and it is more natural to simply consider

$$\mathrm{Pic}_0(X) \stackrel{\pi_f}{\twoheadrightarrow} E.$$

Cohomology in higher dimension

Replace Shimura curve X by a variety $V_{/F}$, $\mathrm{char}(F) = 0$, of dimension $d \geq 1$.

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For curves: $\mathrm{Fil}^0 = H^1_{dR}(X) = \Omega^{II}(X)/dF(X) \supset \mathrm{Fil}^1 = \Omega^1(X)$.

Comparison theorems

For any prime p, the p-adic étale cohomology groups

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$F = \mathbb{C}$: $H^n_{dR}(V/\mathbb{C}) = H^n_{Betti}(V(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C} \simeq \oplus_{i+j=n} H^{i,j}(V/\mathbb{C})$

$$\langle \omega_1, \omega_2 \rangle = \frac{1}{(2\pi i)^d} \int_{V(\mathbb{C})} \omega_1 \wedge \omega_2.$$

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Hodge conjecture: cl is surjective.

The complex Abel-Jacobi map

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^1(X)_0 \longrightarrow (H^{1,0})^\vee / H_1(X,\mathbb{Z}), \quad D \mapsto \int_D$$

generalizes:

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$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^c(V)_0(\mathbb{C}) \longrightarrow J^c(V), \qquad \Delta \mapsto \int_{\partial^{-1}\Delta}$.

$\tilde{\Delta} = \partial^{-1}\Delta$ is a $2(d-c)+1$-differentiable chain on the real manifold $V(\mathbb{C})$ with boundary Δ.

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Tate: there is $\Pi^? \in \mathrm{CH}^{d+1-c}(V \times E)(\mathbb{Q})$ inducing

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$$\Delta \in \mathrm{CH}^c(V)_0 \mapsto \pi_V^*\Delta \mapsto \pi_V^*\Delta \cdot \Pi^? \mapsto P_\Delta := \pi_{E,*}(\pi_V^*\Delta \cdot \Pi^?) \in E$$

Chow-Heegner points

Thus also want "non-trivial looking" null-homologous cycles

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Shimura varieties associated to a reductive group $G_{/\mathbb{Q}}$ host special cycles.

Example 1: modular and Shimura curves

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\begin{array}{ccc}
\mathrm{CH}^1(V)_0(\mathbb{C}) & \overset{\mathrm{AJ}_\mathbb{C}}{\rightarrow} & \mathrm{Jac}(V) \\
\pi \downarrow & & \downarrow \pi_\mathbb{C} \\
E(\mathbb{C}) & \overset{\mathrm{AJ}_\mathbb{C}}{\rightarrow} & \mathbb{C}/\Lambda_E,
\end{array}
$$

$$D = ([\tau] - \infty) \in \mathrm{CH}^1(V)_0 \mapsto P_D \in E.$$

Example 2: Kuga-Sato varieties

The universal elliptic curve is

$$\pi : V_1 \twoheadrightarrow X_1(N)$$

with generic fiber $\pi^*(x) = E_x$, an elliptic curve with a N-torsion point t_x.

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The approach of M. Bertolini, H. Darmon and K. Prasanna:

$$S_{r+2}(\Gamma_1(N)) \simeq \varepsilon H_{par}^{r+1,0}(V_r), \quad f(q) \mapsto f(q) dz_1 ... dz_r dq/q.$$

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Say $D = 11, 19, 43, 67, 163$, so $\mathbb{Q} \otimes E(\mathbb{Q}) = \mathbb{Q} \cdot P_E$.

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$\overset{\text{Tate?}}{\Rightarrow} \quad \Pi^? \in \mathrm{CH}^{r+1}(E^{r+1} \times V_r)(K)$

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X_r has dimension $2r+1$ and hosts Heegner cycles of codimension $r+1$.

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Numerically found that for odd r:

$$P_{r,\mathbb{C}} = \sqrt{-D}\cdot m_r \cdot P_E, \quad m_r^2 = \frac{2r!(2\pi\sqrt{D})^r}{\Omega_E^{2r+1}} L(\psi_E^{2r+1}, r+1) \in \mathbb{Z}.$$

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And proved a p-adic étale version of this.

Let $E_{/\mathbb{Q}}$ be an arbitrary elliptic curve, of conductor N.

Example 3: triple products

Let $E_{/\mathbb{Q}}$ be an arbitrary elliptic curve, of conductor N.

Let $V_{/\mathbb{Q}} = V_r \times V_r \times X, \quad X = X_0(N)$ or $X_0^{N^-}(N^+)$.

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It yields

$$\begin{array}{cccccc} \pi : \mathrm{CH}^{r+2}(V)_0 & \rightarrow & \mathrm{Pic}_0(X) & & \stackrel{\pi_f}{\rightarrow} E \\ \Delta & \mapsto & P_\Delta & = & \sum_{(P,P,Q)\in\Delta} \pi_f(Q) \end{array}$$

$\dim(V) = 2r + 3$ and there are several natural choices for

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For $r \geq 1$, $\Delta_r := (\epsilon, \epsilon, \mathrm{Id})(\Delta_{\{1,2,3\}} - \Delta_{\{1,2\}}) \in \mathrm{CH}^{r+2}(V)_0$

Theorem (Darmon-R-Sols) $P_r := P_{\Delta_r} \in E(\mathbb{Q})$ satisfies

$$P_r = n_r P_0, \quad n_r \in \mathbb{Z},$$

with $P_0 = \pi_{E,*}(K_X)$, where $K_X \in \mathrm{Pic}(X)$ is the canonical divisor.

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Theorem (Yuan-Zhang-Zhang) $P_g \neq 0$ in $\mathbb{Q} \otimes E(\mathbb{Q}) \Leftrightarrow$

$$\mathrm{ord}_{s=1} L(E,s) = 1 \text{ and } L(E \otimes \mathrm{sym}^2(g), 2) \neq 0.$$