Stark-Heegner Points

Henri Darmon and Victor Rotger

Second Lecture

Arizona Winter School

Tucson, Arizona

March 2011
Summary of the first lecture

Victor described variants of the Heegner point construction based on higher dimensional algebraic cycles: the so-called *Chow-Heegner points*.

Our last two lectures, and the student projects will focus *exclusively* on Chow-Heegner points attached to *diagonal cycles* on triple products of modular curves and Kuga-Sato varieties.

**Goal of this morning’s lecture**: indicate how these ostensibly very special constructions fit into the “broader landscape” of *Stark-Heegner points*. 
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otherwise the less experienced participants might feel like the protagonists in the tale of the elephant and the six blind men!
What is a Stark-Heegner point?

Executive summary: Stark-Heegner points are points on elliptic curves arising from (not necessarily algebraic) cycles on modular varieties.
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Motivation. Thanks to Heegner points, we know:

$$\text{ord}_{s=1} L(E, s) \leq 1 \implies \text{rank}(E(\mathbb{Q})) = \text{ord}_{s=1} L(E, s),$$

for all elliptic curves $E/\mathbb{Q}$. (Gross-Zagier, Kolyvagin.)

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The mysterious elliptic curves

$F =$ a real quadratic field;

$E =$ elliptic curve of conductor 1 over $F$;

$\chi : \text{Gal}(K/F) \longrightarrow \pm 1 =$ quadratic character of $F$.

**Question:** Show that

$$\text{ord}_{s=1} L(E/F, \chi, s) \leq 1 \implies \text{rank}(E^\chi(F)) = \text{ord}_{s=1} L(E/F, \chi, s).$$
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**Theorem (Matteo Longo)**

\[
L(E/F, \chi, 1) \neq 0 \implies \#E^\chi(F) < \infty.
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Yu Zhao’s PhD thesis (defended March 10, 2011):

**Theorem (Rotger, Zhao, D)**

If \( E \) is a \( \mathbb{Q} \)-curve, i.e., is isogenous to its Galois conjugate, then

\[
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We have no idea how to produce a point on \( E^\chi(F) \) in general!

**Logan, D, (2003):** We can nonetheless propose a *conjectural formula* to compute it in practice, via *ATR cycles.*
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**Logan, D, (2003):** We can nonetheless propose a *conjectural formula* to compute it in practice, via *ATR cycles.*
Let $Y$ be the (open) *Hilbert modular surface* attached to $E/F$:

$$Y(\mathbb{C}) = \text{SL}_2(\mathcal{O}_F) \backslash (\mathcal{H}_1 \times \mathcal{H}_2).$$

Let $\gamma \in \text{SL}_2(\mathcal{O}_F)$, with a (unique) fixed point $\tau_1 \in \mathcal{H}_1$.

Then the field $K$ generated by the eigenvalues of $\gamma$ is an ATR extension of $F$.

To each $\gamma$, we will attach a cycle $\Delta_\gamma \subset Y(\mathbb{C})$ of real dimension one which “behaves like a Heegner point”.
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[Definition: ATR cycles]

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\[ \tau_2, \tau'_2 := \text{fixed points of } \gamma \circ (\mathcal{H}_2 \cup \mathbb{R}); \]

\[ \mathcal{Y}_\gamma = \{\tau_1\} \times \text{geodesic}(\tau_2 \to \tau'_2). \]

\[ \Delta_\gamma = \mathcal{Y}_\gamma/\langle \gamma \rangle \subset \mathcal{Y}(\mathbb{C}). \]

**Key fact:** The cycles \( \Delta_\gamma \) are null-homologous.
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Oda’s conjecture on periods

For any closed 2-form $\omega_G \in \Omega_G$, let $\Lambda_G$ denote its set of periods, as in Kartik’s lectures:

$$\Lambda_G = \{ \int_\gamma \omega_G, \quad \gamma \in H_2(X(\mathbb{C}), \mathbb{Z}) \}.\]
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Conjecture (Oda, 1982)

For a suitable choice of $\omega_G$, we have $\mathbb{C}/\Lambda_G \sim E(\mathbb{C})$. 
Points attached to ATR cycles

\[ P_\gamma^2(G) := AJ(\Delta_\gamma)(\omega_G) := \int_{\partial^{-1}\Delta_\gamma} \omega_G \in \mathbb{C}/\Lambda_G = E(\mathbb{C}). \]

\[ \Gamma_{\text{trace} = t} = (\Gamma_{\gamma_1}\Gamma^{-1}) \cup \cdots \cup (\Gamma_{\gamma_h}\Gamma^{-1}). \]

Conjecture (Logan, D, 2003)

The points \( P_\gamma^{\gamma_j}(G) \) belongs to \( E(H) \otimes \mathbb{Q} \), where \( H \) is a specific ring class field of \( K \). They are conjugate to each other under \( \text{Gal}(H/K) \), and the point \( P_K^2(G) := P_{\gamma_1}^2(G) + \cdots + P_{\gamma_h}^2(G) \) is of infinite order if and only if \( L'(E/K, 1) \neq 0 \).
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ATR points are defined over abelian extensions of a quadratic ATR extension $K$ of a real quadratic field $F$.

There is a second setting, equally fraught with mystery, involving an elliptic curve $E/\mathbb{Q}$ over $\mathbb{Q}$ and class fields of \textit{real quadratic fields}.

**Simplest case:** $E/\mathbb{Q}$ is of prime conductor $p$, and $K$ is a real quadratic field in which $p$ is inert.

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1. Attaching to \( f \in S_2(\Gamma_0(p)) \) a “Hilbert modular form” \( G \) on \( \text{SL}_2(\mathbb{Z}[1/p]) \backslash (\mathcal{H}_p \times \mathcal{H}) \).

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\int_{\partial^{-1} \Delta_\gamma} \omega_G \in K_p^\times / q^\mathbb{Z} = E(K_p)
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for any “\( p \)-adic ATR cycle” \( \Delta_\gamma \).

The resulting local points are defined \((\textit{conjecturally})\) over ring class fields of \( K \). They are prototypical “Stark-Heegner points” …
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Computing Stark-Heegner points attached to real quadratic fields

There are fantastically efficient *polynomial-time* algorithms for calculating Stark-Heegner points, based on the ideas of Glenn Stevens and Rob Pollack. (Cf. their AWS lectures.)
Drawback of Stark-Heegner vs Chow-Heegner points

They are completely mysterious and the mechanisms underlying their algebraicity are poorly understood.
Advantage of Stark-Heegner vs Chow-Heegner points

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New cases of the BSD conjecture?

Theorem (Bertolini, Dasgupta, D + Longo, Rotger, Vigni)

Assume the conjectures on Stark-Heegner points attached to real quadratic fields (in the stronger, more precise form given in Samit Dasgupta’s PhD thesis). Then

\[ L(E/K, \chi, 1) \neq 0 \implies (E(H) \otimes \mathbb{C})^{\chi} = 0, \]

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Question. Can we control the arithmetic of \( E \) over ring class fields of real quadratic fields without invoking Stark-Heegner points?
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**Question.** Can we control the arithmetic of \( E \) over ring class fields of real quadratic fields *without* invoking Stark-Heegner points?
Diagonal cycles on triple products of Kuga-Sato varieties

Let \( r_1 \geq r_2 \geq r_3 \) be integers, with \( r_1 \leq r_2 + r_3 \).

\[ r = \frac{r_1 + r_2 + r_3}{2} \]

\[ V = E^{r_1} \times E^{r_2} \times E^{r_3}, \quad \text{dim } V = 2r + 3. \]

\[ \Delta = E^r \subset V. \]

\[ \Delta \in CH^{r+2}(V). \]

\[ \text{cl}(\Delta) = 0 \text{ in } H_{et}^{2r+4}(V_{\overline{Q}}, \mathbb{Q}_\ell)(r + 2)^{G_{\mathbb{Q}}}. \]

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Diagonal cycles and $L$-series

Let $f, g, h$ be modular forms of weights $r_1 + 2$, $r_2 + 2$ and $r_3 + 2$.

By taking the $(f, g, h)$-isotypic component of the class $\text{AJ}_{et}(\Delta)$, we obtain a cohomology class

$$\kappa(f, g, h) \in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(r + 2))$$

Its behaviour is related to the central critical derivative

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We don’t “really care” about these rather recundite $L$-series with Euler factors of degree 8...
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Write $\chi = \chi_1 \chi_2$, where $\chi_1$ and $\chi_2$ are characters of signature $(1, -1)$, so that

$$V_1 = \text{Ind}_F^\mathbb{Q} \chi_1, \quad V_2 = \text{Ind}_F^\mathbb{Q} \chi_2$$

are odd two-dimensional representations of $\mathbb{Q}$.

**Hecke:** There exists modular forms $g$ and $h$ of weight one, such that

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Furthermore,

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Hida families

A slight extension of what we learned in Rob’s lecture:

**Theorem (Hida)**

There exist $q$-series with coefficients in $\mathcal{A}(U)$,

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\begin{align*}
g &= \sum_{n=1}^{\infty} b_n(k) q^n, \\
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**Philosophy:** The natural $p$-adic invariants attached to (classical) modular forms varying in $p$-adic families should also vary in $p$-adic families.

**Example:** The Serre-Deligne representation $V_g$ of $G_{\mathbb{Q}}$ attached to a classical eigenform $g$.

**Theorem**

There exists a $\Lambda$-adic representation $V_g$ of $G_{\mathbb{Q}}$ satisfying

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For each $k \in \mathbb{Z}^{>1}$, consider the cohomology classes

$$\kappa_k := \kappa(f, g_k, h_k) \in H^1(\mathbb{Q}, V_f \otimes V_{g_k} \otimes V_{h_k}(1)).$$

**Conjecture**

There exists a “big” cohomology class $\kappa \in H^1(\mathbb{Q}, V_f \otimes V_{g} \otimes V_{h}(1))$ such that $\kappa(k) = \kappa_k$ for almost all $k \in \mathbb{Z}^{\geq 2}$.

**Remark:** This is in the spirit of work of Ben Howard on the “big” cohomology classes attached to Heegner points.

**Question:** What relation (if any!) is there between the class

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Before seriously attacking the study of $p$-adic deformations of diagonal cycles and their (eventual) connection with Stark-Heegner points, it is natural to make a careful study of diagonal cycles and their arithmetic properties.

Because of our predilection for the BSD conjecture—and because elliptic curves are most amenable to computer calculation—we are interested in settings where these diagonal cycles give rise to Chow-Heegner points on elliptic curves, as described in Victor’s first lecture.

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