\[ f \in S_2 \left( \Gamma_0(N) \right) \text{ newform} \mapsto E. \]

\[ B \text{ quaternion indet./\@/@ @ disc } N | N. \]

\[ N = N^+ N^- \]

\[ X_B \text{ associated to an (Eichler) order of level } N^+ \text{ in } B. \]

\[ \langle f_B, f_B \rangle \langle f, f \rangle \quad (f_B, f \text{ normalized up to } \ell \text{-adic units}). \]

Assume \( P_{+, \omega} \) is irreducible. (\( \omega \) is not Eisenstein).

\[ \frac{\langle f, f \rangle}{\langle f_B, f_B \rangle} \cong \prod_{C \subset \mathbb{P} \mathcal{L} \mathcal{B}} \frac{\prod_{C_p}}{\prod_{\mathcal{P} N^-}} \]

True for abelian variety quotients.

\[ \langle f_B, f_B \rangle \cong \frac{\langle f, f \rangle}{\prod_{\mathcal{C} \subset \mathcal{P} \mathcal{L} \mathcal{B}}} = \frac{L(1, \text{ad}^f)}{\prod_{C_p} \frac{\prod_{\mathcal{C} \subset \mathcal{P} \mathcal{L} \mathcal{B}}}{\mathcal{P} \mathcal{L} \mathcal{B}}} \]
$F = \text{tot real.}$

$\exists \ C_v, \rightarrow C_{v_d}, \text{ s.t.}$

$\langle f_{B^i}, f_{B^j} \rangle \sim \prod C_{v_i} \sim \frac{\prod C_{v_i}}{\prod C_{v_i}} \sim \frac{\prod C_{v_i}}{\prod C_{v_i}}$

$B \text{ split at } v_i$

$B \text{ ram at } v_i$

Assume $A$ is not Ei$_d$ for $f$.

Conj: $\exists$ a function: $C : \Sigma(\pi) \rightarrow C^*$, st.

$\langle f_{B^i}, f_{B^j} \rangle \sim \text{a-units} \quad \frac{L(1, ad^*f)}{\prod C_v}$

$\text{ for } v \in \Sigma_{B}$

- If $v$ is inf, expect $C_v$ are transcendental, and alg. ind. except if $f$ is a Base change.
- If $v$ is finite, expect $C_v$ are ($\mathbb{A}$-adic) integers, & count level-lowering congruences.
Recall Conjecture:

\[ F \text{ tot real, } f \text{ HM newform, } \pi = \text{aut rep} \]

\( \exists \) invariants \( C_v, \; \forall \in \Sigma(\pi) \), such that

\[
\langle f_B, f_B \rangle = \frac{L(1, ad^0 \pi)}{\prod \limits_{\forall \in \Sigma(B)} CV} \quad \text{(up to } \pi \text{ is prime)}
\]

- If \( \nu \) is infinite, \( C_v = \text{transcendental} \)
- If \( \nu \) is finite, \( C_v = \text{algebraic integer} \)

\( (\text{if } p \mid CV, \text{then } f \equiv g (p), \forall \text{ level}(g)) \)

Notes: Thm. Suppose \( F = \mathbb{Q}, f \leftrightarrow \text{isogeny class of elliptic curve} \)

\( f \in \mathcal{S}_2(\Gamma_0(N)), N \text{ square-free} \).

What are the \( C_v \)'s? \( \Sigma(\pi) = \{ \infty \} \cup \{ q \mid q \mid N \} \).

\[
C_{\infty} = \sum \limits_{E \in \mathcal{W}_E \wedge \overline{W}_E} \text{WE} \text{ any elliptic curve in isogeny class, } \text{WE} = \text{Neron differential}
\]

For \( q \mid N \), \( C_q = \text{order of component q} \text{ of Neron model of } E \text{ at } q \).

If \( E' \) is another elliptic curve in same isogeny class, \( E \to E' \).
B definite. \((X_B = \text{finite set of pts})\).

\[
\langle f_B, f_B \rangle = \frac{L(1, \alpha d)}{C_0 \cdot \frac{\pi C_q}{g_1 d_B}}
\]

There are relations between \(\langle f_B, f_B \rangle\), as \(B\) varies.

(Student Project 1: Compute this for totally definite \(q\)-algebras \(\mathfrak{G}\))
higher weight

tot real fields

Theta correspondence:

\[ F = \text{number field}, \quad \mathbb{A}_F; \quad W \text{ a symplectic space over } F. \]

\[ \langle , \rangle : W \times W \to F \quad \text{that is nondegenerate,} \]
\[ \text{and alternating}. \]

Fix \( \psi \) an additive character of \( \mathbb{F}^\times : \mathbb{A}_F. \)

Let \( \text{Sp}(W) \) be the symplectic group of \( W. \)

\( \text{GSp}(W) : \text{similitude group} \)

\[ 1 \to \mathbb{C}^\times \to \text{Mp}(W)(\mathbb{A}) \to \text{Sp}(W)(\mathbb{A}) \to 1 \]

Weil \( \Psi \) representation: \( \psi : \text{Mp}(W)(\mathbb{A}) \to \text{Aut}(\Psi) \)

Dual reductive pair: \( \text{(Homr)} \)

\( (G_1, G_2) \) of reductive groups, \( G_1 \times G_2 \subseteq \text{Sp}(W), \)
\( G_1 \) and \( G_2 \) are centralizers of each other in \( \text{Sp}(W). \)
\[ G_1 \times G_2 \subseteq \text{Sp}(\mathcal{W}) \]

\[ \overline{G_1(A)} \times \overline{G_2(A)} \subseteq \left( \overline{\text{MP}_\psi(\mathcal{W}(A))} \right) \]

\[ \overline{G_1(A)} \times \overline{G_2(A)} \subseteq \text{Sp}(\mathcal{W})(A) \]

Can use this to transfer functions from \( G_1(A) \) to \( G_2(A) \) & in the other direction.

**Eg. 1.** \( W = \text{symplectic space}, \ V = \text{orthogonal space}. \) \( W = W \otimes V \) is a symplectic space.
\[ (\text{Sp}(\mathcal{W}), \ O(V)) \] is a dual reductive pair in \( \text{Sp}(W) \).

**Eg. 2.** \( K \)

\[ G/\text{quadratic extn.} \]
\( F \)

\( V_1, V_2 \) unitary spaces \( /K \).

\( V_1: \) \( K \)-vector space, \( \langle \cdot, \cdot \rangle: V_1 \times V_1 \rightarrow K \)

\[ \langle \alpha x, \beta y \rangle = \overline{\alpha} \langle x, y \rangle \beta \]

\[ \langle x, y \rangle = \overline{\langle y, x \rangle} \]

\( V_2: \) \( \langle x, y \rangle = -\overline{\langle y, x \rangle} \).
\[ W = V_1 \otimes_k V_2 \text{ thought of as an } F\text{-vector space.} \]

\[ \langle , \rangle = \text{tr}_K (\langle , \rangle \otimes \langle , \rangle) \text{ skew-symmetric.} \]

\[ (U_K(V_1), U_K(V_2)) \subseteq \text{Sp}(W) \text{ is a dual reductive pair.} \]

---

We can use Weil rep to construct an integrating kernel.

\[ \varphi \in \mathcal{B} \mapsto \Theta_\varphi (b_1, b_2) \]

\[ (b_1, b_2) \subseteq \text{Sp}(W) \cdot \]

\[ f_1 \text{ on } G_{\text{st}}(A) : \quad \Theta_\varphi (f_1) = \int f_1 (b_1) \cdot \Theta_\varphi (b_1, b_2) \, \text{d}b_1. \]

---

**Eq.** (Shimizu correspondence) Bquat alg/F.

\[ V = B, \quad \langle x, y \rangle = xy^i + yx^i, \quad i = \text{main involution.} \]

\[ W = 2\text{-dim symplectic space} \]

\[ (\text{Sp}(W), O(V)) \quad (G_{\text{Sp}}(W), G_0(V)) \]

\[ (\text{GL}_2, (B^x \times B^x) / F) \]

\[ F^+ / B^x \times B^x \to G_0(V)^0 \quad (\alpha, \beta) \mapsto (x \mapsto \alpha x \beta^{-1}). \]
Forms in \((B_1^x \times B_2^x)/F^x\) look like pairs \((\pi_1, \pi_2)\), s.t. \(\omega_{\pi_1}, \omega_{\pi_2} = 1\). Central chars.

\(\pi\) on \(GL_2 \mathbb{A}\); \(\Theta(\pi) = \left\{ \begin{array}{ll} 0 & \text{if } \pi \text{ doesn't transfer to } B^* \\ \pi_B \times \pi_B^\vee, \hspace{1em} \pi_B = JL(\pi) & \end{array} \right. \)

In our case, central chars trivial \(\Rightarrow \pi_B^\vee \cong \pi_B\).

Pick \(f \in \pi\), \(\Theta_\Phi(f) = (\Phi)(f_B \times f_B)\) (can pick \(\Phi\) to make this happen).

\(GL_2 \rightarrow (B_1^x \times B_2^x)/F^x\)

\(GL_2 \leftarrow (B_1^x \times B_2^x)/F^x\) (Easier to study). Since you can compute explicitly with F.C.'s in left.

One can show; \(\beta = \langle f_B, f_B \rangle\).

\(\alpha \langle f_B \times f_B, f_B \times f_B \rangle = \beta \cdot \langle f, f \rangle = \langle f_B, f_B \rangle \langle f, f \rangle\)

\(\alpha = \frac{\langle f, f \rangle}{\langle f_B, f_B \rangle}\)
Seesaw Dual Reductive Pair (Kudla).

\((G_1, G_2), (H_1, H_2) \subseteq \text{Sp}(W)\).

These form a seesaw.

\[
\int f_1 \cdot \Theta_\varphi(f_2) \big|_{H_1} = \int_0 \Theta_\varphi(f_1) \big|_{G_2} f_2 \quad \text{seesaw duality.}
\]

\[E_{\varphi} \quad V \text{ orthogonal, } W \text{ symplectic. } \|W = V \oplus W.
\]

\((O(V), \text{Sp}(W)) \subseteq \text{Sp}(W)\)

\[V = V_1 \oplus V_2 \quad \text{(sum of two orthogonal spaces)}.
\]

\[
\begin{array}{c}
\text{Sp}(W) \times \text{Sp}(W) \\
\text{Sp}(W) \quad O(V_1) \quad O(V_1) \times O(V_2)
\end{array}
\]
\[(B^* x B^*)/F^* = G_0(V) \Rightarrow \mathfrak{O}(f B \times f B) \]

\[GL_2 f\]

\[F = \mathbb{Q}, \quad B \text{ indefinite.}\]

Form on \(B^* \times B^* \) section of a line bundle on \(X_B\).

(Usual Modular forms: function on pairs \((E, w)\).)

\(X_B: \text{ coarse moduli space, abelian surface with end. by } B\).

Check this on CM points: \(K \rightarrow B, \ K^x \rightarrow B^x \)

\[L_x(g) = \int_{K^x/A} g \cdot x \quad \text{Pick a Hecke character } \chi \text{ of type } (2, 0) \quad \text{weight.}\]

= "finite sum of values of } g, \text{ twisted by } x".

Criterion: } g \text{ is rational (integral) if } L_x(g) \text{ are rational (integral) up to periods of CM elliptic curve. (CM periods).}
\[ GL_2 \times GL_2 \]

\[ G_0(V) = (B_1 \times B_2)/F_1 \]

\[ (x_1 x_2, y_1 y_2) \rightarrow (x_1 y_1, x_2 y_2) \]

\[ k \rightarrow \theta \]

\[ V = V_1 \oplus V_2 \]

\[ \alpha \cdot \int f_0(x_1 y_1) \cdot \Theta(x \cdot y) = \int \Theta(x) \cdot \theta(y) \cdot f(y) \cdot y \]

\[ \alpha \cdot \frac{\int x^2 f_0(x)^2}{\int x^2 \cdot \text{r.m.s.}} = \frac{\langle f, \Theta(y), \Theta(x) \rangle}{\sum_{\text{r.m.s.}} \langle f, E(s), \Theta(x) \rangle} \]

\[ \text{value at } s = \frac{1}{2} \]

\[ \text{value of Eisenstein series: } E(5) \]

- Harris-Kudla: L-value is rational.
- P. (2003): L-values are integral (use Main conjecture for imaginary quadratic fields: (Rubin)).
- Factorization: (p-adic families).