Talk 2:

Stable reduction &

necc. conditions for liftability

of covers

$k, \sigma, \bar{k}$ alg. closed of char $p > 0$

v, m

$Y/k \qquad G < \mathrm{aut}_k(Y)$ finite

$\leadsto f: Y \longrightarrow X = Y/G$

$\{y_1, \ldots, y_r\}$ ram. pts

<u>σ-model</u> $\mathcal{Y}/\sigma$ normal, flat, proper
st $\mathcal{Y} \otimes_\sigma k \simeq Y$.

<u>G-semistabel</u>: $\cdot\ \bar{Y} = \mathcal{Y} \otimes_\sigma \bar{k}$

- has ord. double pts as sing
- G-action extends to $\mathcal{Y}$
- y_i specialize to smooth pts $\bar{y}_i \neq \bar{y}_j$

$\rightsquigarrow f_\sigma : \mathcal{Y} \longrightarrow \mathcal{X} = \cancel{\ } \mathcal{Y}/G$.

$\mathcal{X}$ semistable

If $2g(Y) + r - 2 > 0$, then
$\exists!$ minimal G-s.s. called G-stable
after replacing k by a fin. ext.

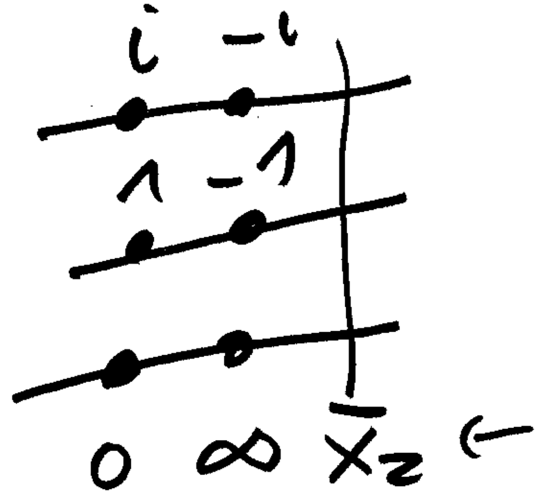
<u>Exa</u> $p = 2$ $\quad k = \mathbb{Q}_2^{ur}[i]$

$$(*) \quad y^2 = x(x^4 - 1) =: g \qquad \begin{array}{c} Y \longrightarrow X \\ (x,y) \longmapsto x \end{array}$$

$\rightsquigarrow \mathcal{Y}_0 \longrightarrow \mathcal{X}_0$

$\overline{Y}_0 \longrightarrow \overline{X}_0$ sing in $\bar{x} = 1$

$\rightsquigarrow$



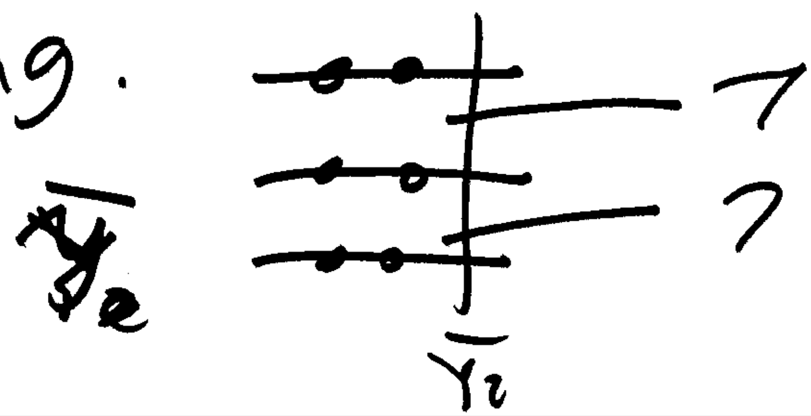
$x = \sqrt{2}\, x_2 + 1$

$\mathcal{X}_2 = \mathbb{P}^1_\sigma$ with coord x_2

$\mathcal{Y}_2$ = Norm of x_2 in $k(Y)_2$

$\bar{Y}_2 : \quad \bar{y}_2^2 = \bar{x}_2 (1 + \bar{x}_2 + \bar{x}_2^2)$

$\rightsquigarrow$ 2 sing.



bad red.

B3

(b) $p > 3$ $\quad (*)\ y^p = x^3 - ax + 1 =: g(x)$

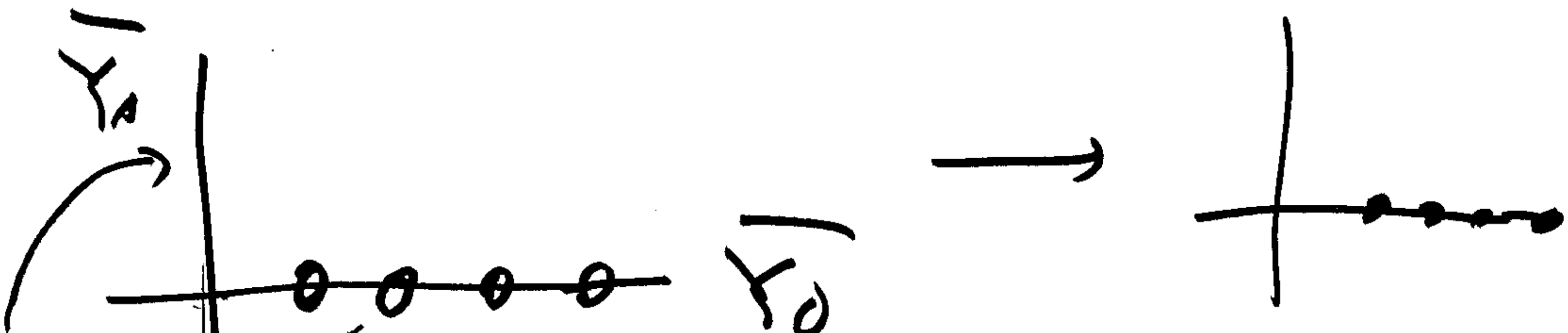
$Y \longrightarrow X, \quad (x,y) \longmapsto x.$

$\rightsquigarrow \mathcal{Y}_0 \longrightarrow \mathcal{X}_0$

Sing. of $\bar{Y}_0$: $g'(x) \equiv 3x^2 - a \equiv 0 (p)$

$\boxed{A}$ $a \not\equiv 0 (p)$: 2 sing. $\rightsquigarrow$ bad red

$\boxed{B}$ $a = p$: 1 sing $\rightsquigarrow$ good red



$\bar{y}^p - \bar{y} = \bar{x}^3$

First Obstruction for liftability:

$\bar{f}: \bar{Y} \to \bar{X}$ G-Galois

$\bar{y} \in \bar{Y}$ ram pt. $G_{\bar{y}}$

$\rightsquigarrow L_{\bar{y}} \to K_{\bar{x}}$ $G_{\bar{y}}$-ext

$G_{\bar{y}} \cong P \rtimes C$.

$\rightsquigarrow G_{\bar{y}} = G_0 \supset G_1 \supset \dots \supset G_{h_r} \supsetneq G_{h_r+1} = \{1\}$

$i_{G_{\bar{y}}}(\sigma) = v(\sigma(\pi_L) - \pi_L)$

$G_t = \{\sigma \in G_{\bar{y}} \mid i_G(\sigma) \geq t+1\}$

Def <u>Artin character</u> of $L_{\bar{y}}/k_{\bar{y}}$

$$a_{\bar{y}}(\sigma) = \begin{cases} -i_G(\sigma) & \sigma \neq 1 \\ \sum_{\sigma \neq 1} i_G(\sigma) & \sigma = 1 \end{cases}$$

class function

<u>Thm</u> (Hasse-Arf-Artin)

$a_{\bar{y}}$ is a character.

Supp $\bar{f}$ lifts: $\exists f_\sigma : \mathcal{Y} \to \mathcal{X}$

st $f_\sigma \otimes_\sigma k$ G-Gal. of smooth

<u>Thm</u> (Bertin Obstruction).

$\bar{y} \in \bar{Y}$ ram. $\exists G_{\bar{y}}$-set $\Delta = \Delta_{\bar{y}}$

st $\chi_\Delta = |\Delta/G| \cdot r_{G_{\bar{y}}} - a_{\bar{y}}$

II-7

Exa

- Exab : $|\Delta| = 4$. fixed by $G \simeq \mathbb{Z}/p\mathbb{Z}$.
- $G = \mathbb{F}_p \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$. + Katz-Gabber.
- BO vanishes iff $h \equiv -1 \pmod{m}$

$$
\begin{array}{ccc}
\overline{Y} & \qquad\qquad & Y \\
\downarrow & & \downarrow \Big\} \ h+1 \text{ branch pts} \\
\overline{Z} = \mathbb{P}^1 & & Z \\
\downarrow & & \downarrow \circlearrowleft \mathbb{Z}/m\mathbb{Z}. \\
\overline{X} = \mathbb{P}^1 & & X
\end{array}
$$

II-8

$G = Q8$ & $\mathrm{char}(k) = 2$

$h = 1 + 2^n$

can show : $\exists\, Y \quad y^2 + y = x^{-h}$

$G = G_0 = G_1 \supsetneq G_2 = \langle -I \rangle = \cdots = G_h \supsetneq \{1\}$

Exc 4.7 BO vanishes

$\Leftrightarrow \quad h \equiv 1 \pmod 4$

$\Leftrightarrow \quad$ ~~$a_0 = 3$~~ $n \geq 2$

Special cases

$\boxed{h = 3 = 1 + 2}$ $\leadsto$ $g(\bar{Y}) = 1$

Supp lifts

$$\begin{array}{c} Y \\ \downarrow \\ \mathbb{P}^1 \\ V \downarrow \\ \mathbb{P}^1 \end{array} \quad \begin{array}{l} 4 \text{ branch pts} \\ 6 \text{ ram. pts.} \end{array}$$

$$\begin{array}{c} \bar{Y} \\ \downarrow \mathbb{Z}/2\mathbb{Z} \\ \mathbb{P}^1 = \bar{Y}/\langle -I \rangle \\ \downarrow V \\ \bar{Y}/Q_8 = \mathbb{P}^1 \end{array}$$

$\lightning$

II-10

$\underline{n=5}$ $\rightsquigarrow$ $g(\overline{Y}) = 2$.

but $\overline{Y} \to \overline{Y}/Q_8 = \mathbb{P}^1$

does not lift.

char 0 $\exists!$ $Q_8 < \mathrm{aut}_k(Y)$:

Y $y^2 = x(x^4 - 1)$.

& Y has bad red.

I/11