Addendum to yesterday's lecture

$PSL_2(\mathbb{Z}) \twoheadrightarrow \underline{\underline{A_6}}$ is noncongruence.

$$A_5 \cong PSL_2(\mathbb{F}_5) \,!$$

Generating S-arithmetic groups by small subgroups

T. Chinburg + M. Stover

B = central division algebra over a number field k

Fact: B may contain many division subalgebras B_F over subfields $F \subseteq k$.

Idea: Try to generate arithmetic subgroups $\Gamma \subseteq B^*$ by the collection of subgroups $\{\Gamma \cap B_F^*\}_{B_F \neq B}$

Geometry: Shimura subvarieties of Shimura varieties

§ 1. Algebra

A particular case

$F = \mathbb{Q} \subseteq k = \mathbb{Q}(\sqrt{d})$, $d > 0$

B/k quaternion div. alg.

$\dim_k B = 4$

Prop: $B = k \otimes B_{\mathbb{Q}}$ for a quat. alg $B_{\mathbb{Q}}/\mathbb{Q}$ iff

1) $B_w \cong \mathrm{Mat}_2(k_w)$ if w is a place of k not split over $\mathbb{Q}$

2) $B_w \simeq B_{w'}$ if $w \neq w'$ lie over same place of $\mathbb{Q}$.

Then $\exists$ ∞ many nonisomorphic choices for $B_{\mathbb{Q}}$

<u>Why</u>: $[B] \in Br(k) =$ Brauer gp

$$\begin{tikzcd}
0 \arrow[r] & Br(\mathbb{Q}) \arrow[r] \arrow[d] & \bigoplus_{v \text{ of } \mathbb{Q}} Br(\mathbb{Q}_v) \arrow[r, "\text{sum}"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\cdot [k:\mathbb{Q}] = 2"] & 0 \\
0 \arrow[r] & Br(k) \arrow[r] & \bigoplus_{w \text{ of } k} Br(k_w) \arrow[r, "\text{sum}"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$$Br(\mathbb{Q}_v) = \begin{cases} \mathbb{Q}/\mathbb{Z} & v \text{ finite} \\ \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} & v \text{ real} \\ & \end{cases}$$

$$\begin{tikzcd}
Br(\mathbb{Q}_v) \arrow[d, "\cdot [k_w : \mathbb{Q}_v]"'] \\
Br(k_w)
\end{tikzcd}$$

$$[B] \in Br(k), \quad [B_w] = \begin{cases} 0 & B_w = Mat_2(k_w) \\ \frac{1}{2} & B_w \text{ div alg.} \end{cases}$$

Diagram determines if $[B]$ comes from a $[B_{\mathbb{Q}}] \in Br(\mathbb{Q})$.

$\mathcal{D} = \mathcal{O}_k$ order in B

$\mathcal{D}^1 = \ker(\text{nred}: \mathcal{D}^* \to \mathcal{O}_k^*)$

<u>Problem</u>: Is $\mathcal{D}^1$ generated up to finite index by a finite collection of subgroups

$$\mathcal{D}^1 \cap B_{\mathbb{Q}}^1 = (\mathcal{D} \cap B_{\mathbb{Q}})^1$$

as $B_{\mathbb{Q}}$ ranges over a finite set of div. algebras over $\mathbb{Q}$ with $k \otimes B_{\mathbb{Q}} = B$?

<u>Note</u>: $\mathcal{D} \cap B_{\mathbb{Q}} =$ a $\mathbb{Z}$ order in $B_{\mathbb{Q}}$

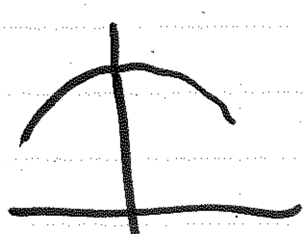
§2. Geometry

$B = k \otimes_{\mathbb{Q}} B_{\mathbb{Q}}$

$\infty_1 \quad \infty_2$ = arch. (real) places of k

∞ = real place of $\mathbb{Q}$

$$\begin{array}{ccc}
(\mathbb{R}\otimes B_{\mathbb{Q}})^1 & \longrightarrow & (\mathbb{R}\otimes B_k)^1 \\
\| & & \| \\
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) & & SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \\
\downarrow \quad \downarrow & \text{tot. geodesic} & \downarrow \\
\gamma . \epsilon \in \mathcal{H} = \dfrac{SL_2(\mathbb{R})}{SO_2(\mathbb{R})} & \longrightarrow & \mathcal{H} \times \mathcal{H} \\
\| & & \| \\
\{ z = x + iy : y > 0 \} & & \dfrac{SL_2(\mathbb{R}) \times SL_2(\mathbb{R})}{\pi_1 SO_2(\mathbb{R}) \times \pi_2 SO_2(\mathbb{R})}
\end{array}$$



$$\frac{\sqrt{dx^2 + dy^2}}{y}$$

Now take $\underset{\text{tors. free}}{\Gamma} \underset{\text{finite index}}{\subseteq} \mathcal{O}^1$

$k \otimes B_{\mathbb{Q}} = B$

Fuchsian Subgroup

$$\Gamma' = \Gamma \cap \mathcal{O}_{\mathbb{Q}}^1 \subseteq \Gamma$$

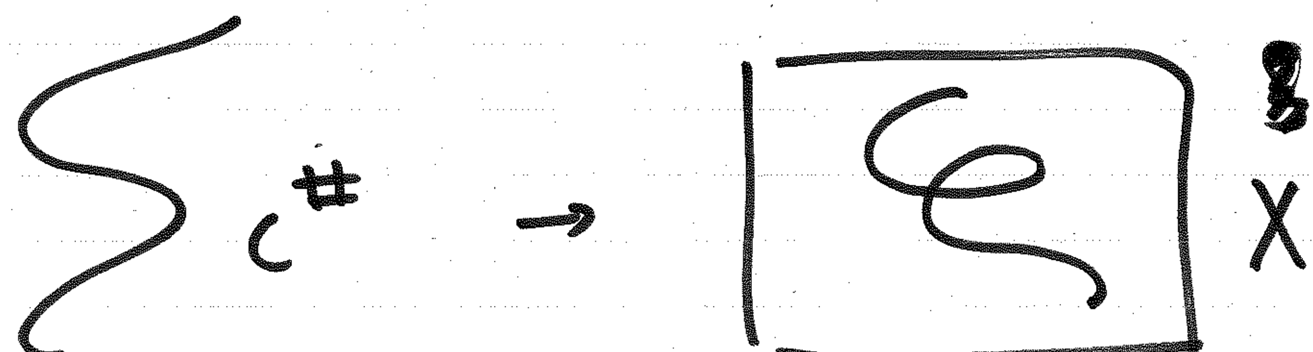
when $\mathcal{O}_{\mathbb{Q}} = \mathcal{O} \cap B_{\mathbb{Q}} = \mathbb{Z}$ order in $B_{\mathbb{Q}}$.

Get:

$$C^{\#} = \Gamma' \backslash \mathfrak{H} \longrightarrow X = \Gamma \backslash (\mathfrak{H} \times \mathfrak{H})$$

compact Shimura curve

Compact Shimura surface



The surface X contains infinitely many Fuchsian curves

$$C = \text{Image}(C^{\#} \longrightarrow X)$$

<u>Problem</u>: Is $\pi_1(X)$ generated up to finite index by the images of $\pi_1(C^{\#})$ as $C^{\#}$ ranges over finitely many curves as above.

We'll use Lefschetz theorems to show a ~~weaker~~ weaker result.

Albanese Variety of X:

$$X \to \mathrm{Alb}(X) \quad \text{universal for morphisms to abelian varieties}$$

$$\mathrm{Alb}(X) = \frac{\mathrm{Hom}_{\mathbb{C}}(H^0(X,\Omega^{1,0}),\mathbb{C})}{\text{period lattice}}$$

$$\Delta \longrightarrow \left(\int_{\Delta_0}^{\Delta}\omega_1,\ldots,\int_{\Delta_0}^{\Delta}\omega_n\right) \bmod \text{periods}$$

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Alb}(X) \\ \uparrow & \nearrow & \uparrow \\ C^{\#} & \longrightarrow & \mathrm{Jac}(C^{\#}) \end{array} \quad \text{universal}$$

$$\mathrm{Jac}(C^{\#}) = \mathrm{Alb}(C^{\#})$$

One can construct more X using

$H^2_{\mathbb{C}}$ = complex hyperbolic plane
= $P(V_-)$
= h negative lines in $\mathbb{C}^3$

with $h : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}$
hermitian of signature $(2,1)$

Choose K/K_0 = CM ext. of no. flds
with $K_0 \neq \mathbb{Q}$

$h : K^3 \times K^3 \to K$ hermitian,
indefinite over one place of K_0

$$\underset{\text{tors. free}}{\Gamma} \ \underset{\text{finite}}{\subseteq} \ SL_3(\mathcal{O}_K) \cap \underset{\text{unitary gp. of } h}{SU(h)}$$

$X = \Gamma \backslash H^2_{\mathbb{C}}$
= compact Shimura surface
with ∞ many Fuchsian curves

<u>Theorem</u> (C+S) Suppose $X = \Gamma \backslash \mathfrak{H} \times \mathfrak{H}$ or $X = \Gamma \backslash H^2_{\mathbb{C}}$ as above. There are finitely many Fuchsian curves $C_i^\# \to C_i$ on X such that

$$\begin{matrix} C_i^\# & \to & C_i \\ & & \cap \\ & & X \end{matrix}$$

$$\prod_i \mathrm{Jac}(C_i^\#) \longrightarrow \mathrm{Alb}(X)$$

is <u>surjective</u>.

What about generating B^* by elements in B_Q^*'s?

<u>Fact</u>: Every $\gamma \in B^1$ can be written as $\gamma = \gamma_1 \gamma_2$ where $\gamma_i \in (B_{Q,i})^*$ for $i = 1, 2$.