Small generators for S-arithmetic groups IV

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<u>Setup</u>: $B = k \otimes_{\mathbb{Q}} B''$, $B_{\mathbb{R}} \cong M_2(\mathbb{R}) \times M_2(\mathbb{R})$,

$B''_{\mathbb{R}} \cong M_2(\mathbb{R})$

(*) $\mathrm{Ram}_f(B) \neq \emptyset$

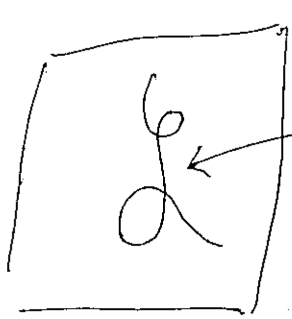
$\Rightarrow$ Shimura curves $\Gamma\backslash\mathbb{H}\times\mathbb{H} \supseteq \Gamma''\backslash\mathbb{H}$ on a Shimura surface

$\mathcal{O}'(3)$ = level 3 congruence subgroup (Borel)

Γ = torsion-free subgroup of $\mathcal{O}' \subset B'$, $\Gamma'' = \Gamma \cap B''$.

<u>Fact</u>: $S = \Gamma\backslash\mathbb{H}\times\mathbb{H}$ is <u>compact</u> by (*). (smooth by t.f.)

$\Rightarrow$ base changes $B'' \to B$ determine curves on S, possibly with self-intersections.



S

$C = \Gamma''\backslash\mathbb{H}$ smooth, projective genus ≥ 2.

<u>Question 1</u>: Can we understand $\mathcal{O}^i$ by the unit groups of orders of the subalgebras $B'' \subset B$ so $B = k \otimes_{\mathbb{Q}} B''$?

<u>Question 2</u>: Can we understand $S = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ via the geometry of the Shimura curves on S?

<u>Note</u>: All our results also hold for $U(2,1)$ Shimura varieties containing $U(1,1)$ Shimura curves.

This is where the Albanese application is most interesting because $\mathrm{Alb}(\Gamma \backslash \mathbb{H} \times \mathbb{H}) = 0$.

<u>Theorem</u> There exists a finite collection of Shimura curves $C_1, \ldots, C_r = \Gamma_j \backslash \mathbb{H}$ on the surface $S = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ or $\Gamma \backslash \mathbb{H}^2_{\mathbb{C}}$ such that

(i) if $\pi_j : C_j \to S$ is the inclusion, the effective divisor $D = \sum_1^r \pi_i(C_i)$ is <u>connected</u>

(2) for all open neighborhoods U of D, the map $\pi_1(U) \to \pi_1(S)$ has finite index in $\pi_1(S)$.
(i.e., $\pi_1(U)$ generates a finite étale covering of S).

- Proof is by weak Lefschetz theory (goes back to Nori's resolution of Zariski's conjecture).

- We will see that in the case of $\mathbb{H} \times \mathbb{H}$, the number of curves is explicitly computable!
(This, in a sense, uses the fact that $Alb(S) = 0$.)

- Analogy with 2D class field theory, understanding a 2D scheme by patching 1D schemes.

<u>Lemma</u> Shimura curves C that are <u>embedded</u> on S are negative.

<u>Idea of proof</u>: Want degree of normal bundle.

Can conjugate B'' in B so it is Galois stable $/\mathbb{Q}$

$\Rightarrow$ $B'' \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \longrightarrow B \otimes_{\mathbb{R}} \mathbb{R} \cong M_2(\mathbb{R}) \times M_2(\mathbb{R})$ is the <u>diagonal map</u> $\Rightarrow$ $\mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{H}$ is diagonal $\Rightarrow$ normal bundle is $\{(v,-v) : v \in T\mathbb{H}\}$ $\Rightarrow$ $\Gamma'' = B'' \cap \Gamma$ action on $N_{\mathbb{H}\times\mathbb{H}}(\text{diagonal}) \cong \Gamma''$-action on $T\mathbb{H}$

$\Rightarrow$ negative of degree $\chi(C) = 2 - 2g(C)$. $\square$

<u>Theorem</u> (Napier–Ramachandran / Campana) Let $\hat{S} \xrightarrow{\hat{\pi}} S$ be an étale cover. If $\hat{S}$ supports a <u>compact</u>, connected, effective divisor of positive self-intersection, then $\hat{\pi}$ is <u>finite</u>.

<u>Recall</u>: If C_1, C_2 distinct irreducible, $(C_1, C_2) = \#(C_1 \cap C_2)$ and $(C_i, C_i) = \deg(N_S(C_i))$. Extend linearly to all divisors.

<u>Fact</u>: $(\, , \,)$ only depends on the class in $H^2(S, \mathbb{Q})$.

<u>Corollary</u>: Let D be a connected, effective divisor on S of positive self-intersection. Then if U is any neighborhood of D in S, then $\pi_1(U) \to \pi_1(S)$ has finite index.

<u>Proof</u>: Let $\hat{S} \to S$ be the étale cover associated with the image of $\pi_1(U)$ in $\pi_1(S)$. Then U lifts to $\hat{S}$, so D also lifts, and (D,D) is the <u>same</u>. Thus D is a compact, connected, effective divisor on $\hat{S}$ of positive self-intersection. Theorem $\Rightarrow$ $\hat{S} \to S$ is finite $\Rightarrow$ $\pi_1(U) \to \pi_1(S)$ finite index. $\square$

<u>Theorem</u>: Let S be a smooth projective surface, $\hat{S} \xrightarrow{f} S$ an étale covering with $\hat{S}$ noncompact, $\hat{C} = \sum_j \hat{C}_j \subset \hat{S}$ effective, compact, connected, $\hat{C}_j$ irreducible.

Then either

① $(\hat{C}, \hat{C}) < 0$

② there exists $\hat{C}_1$ precisely supported on $\hat{C}$ ($\Leftrightarrow$ each $\hat{C}_j$ appears with positive multiplicity) with $(\hat{C}_1)^2 = 0$ and for any other $\hat{C}'$ with support on $\hat{C}$, $(\hat{C}')^2 \leq 0$, $(\hat{C}', \hat{C}_1) = 0$, and for all compact connected curves $\hat{D}$ on $\hat{S}$, $|\hat{D}| \cap |\hat{C}| \neq \phi$ or $(\hat{D}, \hat{C}) = 0$.

($\Rightarrow$ $\hat{C}$ irreducible)

<u>Note</u>: This immediately implies the previous theorem, since $(\hat{C}, \hat{C}) > 0$ is not a possibility.

Divisor connected $\Rightarrow$ (,) restricted to the subspace of $H^2(S,\mathbb{Q})$ generated by the irreducible components is indecomposable.

(2) is a geometric consequence of a lemma about indefinite indecomposable quadratic forms due to Coxeter.

<u>Proposition</u>: Let $\mathcal{L}$ be an ample line bundle on S, $f:\hat{S}\to S$ étale, $\hat{S}$ noncompact. Then for $N \gg 0$,

$$H^0_{(2)}(\hat{S}, f^*(K_S \otimes \mathcal{L}^N))$$

($(2) \Rightarrow L^2$-sections) is infinite dimensional.

This is an infinite version of the fact that if f is finite of degree d, $\dim H^0_{(2)}(\hat{S}, f^*(K_S\otimes\mathcal{L}^N)) = d\cdot \dim H^0_{(2)}(S, K_S\otimes\mathcal{L}^N)$ by Kodaira Vanishing.

Lemma Let U be an open neighborhood of a connected compact effective divisor $\hat{D}$ on $\hat{S}$ and $\mathcal{L}$ a line bundle on U. Then $\dim H^0(U, \mathcal{L}) < \infty$.

To prove the theorem, apply the Prop$^{\underline{n}}$ & Lemma to the inverse image of $D \subset U \subset S$ in $\hat{S}$ = the étale covering of S determined by the image of $\pi_1(U) \to \pi_1(S)$.

Back to Shimura curves.

Infinitely many $\Rightarrow$ can find a relation

$$X = \sum_i m_i C_i = Y = \sum_j n_j C_j' \quad \text{in } H^2(S, \mathbb{Q})$$

where $\{C_i\} \cap \{C_j'\} = \phi$

$$\Rightarrow (X, X) = (X, Y) \geq 0.$$

Prove ~~&~~ using Hecke operators that we can find <u>another</u> curve C_0 so $(C_0, X) > 0$, so if $(X, X) = 0$, then there exists $N > 0$ so $C_0 + NX$ is a positive effective curve made entirely from Shimura curves.

The number of curves necessary is computable!

$$\chi(\mathcal{D}' \backslash \mathbb{H} \times \mathbb{H}) = \frac{-\zeta_F(-1)}{2} \prod_{v \in Ram_f(B)} (N(v) - 1) \qquad \text{(Shimizu)}$$

$$= 2 - \underbrace{4h^{1,0}(\mathcal{D}' \backslash \mathbb{H} \times \mathbb{H})}_{\frac{1}{2} b_1} + \underbrace{2h^{2,0}(\mathcal{D}' \backslash \mathbb{H} \times \mathbb{H}) + h^{1,1}(\mathcal{D}' \backslash \mathbb{H} \times \mathbb{H})}_{b_2}$$

Matsushima – Shimura : $h^{1,0}(S) = 0$ for all $S = \Gamma\backslash\mathbb{H}\times\mathbb{H}$ arising from quaternion algebras.

$$\Rightarrow \chi = 2 + 2h^{2,0} + h^{1,1}.$$

Also $\chi(\mathcal{O}_S) = 1 - h^{1,0} + h^{2,0} = \frac{1}{12}\left(c_1^2 + c_2\right)$

$\mathbb{H}\times\mathbb{H} \Rightarrow c_1^2 = 2c_2 \Rightarrow 1 + h^{2,0} = \frac{3}{12}\left(2 + 2h^{2,0} + h^{1,1}\right)$

(Noether)

$$\Rightarrow 4 + 4h^{2,0} = 2 + 2h^{2,0} + h^{1,1}$$

$$\Rightarrow h^{1,1} = 2 + 2h^{2,0}$$

$$\Rightarrow h^{1,1} = \frac{1}{2}\chi$$

$[\mathcal{O}' : \Gamma]$

$\Rightarrow$ take $\frac{-\zeta_k(-1)}{4}\prod_{v\in \mathrm{Ram}_f(B)}(N(v)-1) + 1$ curves.

$\Rightarrow \Gamma$ is generated (up to finite index) by a neighborhood of $-\frac{\zeta_k(-1)}{4} \prod_{v \in Ram_f(B)} (N(v)-1) + 1$ arithmetic Fuchsian subgroups.

$[\mathcal{O}' : \Gamma]$

<u>Question</u>: Can we remove the neighborhood assumption and conclude that $\mathcal{O}'$ has a subgroup of finite index generated by $\mathcal{O}'_1, \ldots, \mathcal{O}'_M$, where $\mathcal{O}'_j = \mathcal{O}' \cap B_j$, $B_j/\mathbb{Q}$, $B \cong k \otimes_{\mathbb{Q}} B_j$, $\mathbb{R} \otimes_{\mathbb{Q}} B_j \cong M_2(\mathbb{R})$?

Final questions

- Lenstra for algebras over characteristic p fields?
  - Can make sense of Minkowski.
  - May be related to questions about FP_m for algebraic groups over function fields.

A group Γ is FP_m/R, R a ring $(\mathbb{Z})$ $\Leftrightarrow$ there exists a projective resolution

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to R \to 0$$

of finitely generated $R\Gamma$-modules.

- $FP_1/\mathbb{Z} \Leftrightarrow \Gamma$ finitely generated
- $FP_2/\mathbb{Z} \Leftarrow \Gamma$ finitely presented.

__Theorem__ (Nagao) $SL_2(\mathbb{F}_p[t])$ is __not__ FP_1.

__Question__: What is $\sup_{m>0}(\Gamma \text{ is } FP_m)$ where Γ is the S-unit group of a division algebra over a function field?

__Remark__ Serre $\Rightarrow$ units are FP_∞.

- Do our results on generating Shimura surfaces by Shimura curves generalize to
  - Surfaces in characteristic p?
  - $U(n,1)$ Shimura varieties defined by hermitian forms / CM fields ($\Rightarrow$ further Albanese structure thms)
  - Unitary Shimura varieties containing subvarieties of __Kottwitz type__ ($\Rightarrow$ strong new vanishing theorems for these varieties)