Division Algebras and Patching

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$F$ a field

Recall: A central simple alg. $\mathbb{F}$
is a finite dimensional associative $F$-algebra with no
nontrivial two-sided ideals and has center $F$.

A division algebra $\mathbb{F}$ is a CSA $\mathbb{F}$
in which every non-zero element has inverse.

$H = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$

$i^2 = j^2 = -1 \quad ij = k = -ji$
#1 has subfields \( \mathbb{R}, \mathbb{C} \)

\[ \text{division}\mathbb{F}, \ a \in \mathbb{D}\backslash\mathbb{F} \]

\[ \mathbb{F} \supseteq \mathbb{F}[a] \supseteq \mathbb{D} \rightarrow \mathbb{F}[a]\] subfield

Natural to study subfields.
Def: $G$ a finite group

We say $G$ is admissible if $\exists$

- a $G$-Galois field extension $E/F$
- an $F$-division algebra $D$ containing $E$

such that $[E:F] = \text{deg}_F(D) = \sqrt{\dim_F(D)}$

Ex: $G = \mathbb{Z}/2\mathbb{Z}$ complex conjugation over $\mathbb{C}/\mathbb{R}$

$\Rightarrow G$ admissible over $\mathbb{R}$. 
Remarks:

1) If ELF is as in Def., then E is a maximal subfield of D

2) If G is admissible as a division alg. over D and Ext. ELF, structure of D can be recovered from E and G.
Def: A finite group. A crossed product algebra $A$ is defined by

- $E$ is a finite $G$-Galois extension
- $A := \bigoplus_{\sigma \in \mathcal{G}} E u_{\sigma} \quad \sigma \in \mathcal{G}
  \quad u_{\sigma} = 1$

- A 2-cocycle $c : G \times G \to E^*$
  \[ \sigma(c(\tau, \rho)) \cdot c(\sigma, \tau \rho) = c(\sigma \tau, \rho) \cdot c(\sigma, \tau) \]
  for all $\sigma, \tau, \rho \in G$
1. \( c \) normalized
\[
c(\lambda, \sigma) = \lambda = c(\sigma, \lambda) \quad \text{all } \sigma \in G
\]

2. Multiplication defined by
\[
\sigma \cdot b = \sigma(b) \cdot u_\sigma \quad \text{all } b \in E, \sigma \in G
\]
\[
u_\sigma \cdot u_\tau = c(\sigma, \tau) u_{\sigma \tau} \quad \text{all } \sigma, \tau \in G
\]

Lemma: A \( G \)-crossed product algebra is a CSA / F.
\[ H = (R \otimes R) \oplus (R \otimes R) j \]

\[ u_\sigma \cdot a = j (\alpha + i\beta) \]

\[ = (\alpha - i\beta) j \]

\[ = \sigma(a) \cdot u_\sigma \]

**Generally:**

\( G \) admissible with division algebra \( D, E \) surject

\( \Rightarrow D \) is \( G \)-crossed product.
Cyclic algebras

A cyclic of order n, $E_1F$ a $G$-Cohn extension, $G = \langle \sigma \rangle$, $\alpha \in F^n$

$0 \leq i, j \leq n - 1$

$$\delta_{\sigma, \alpha} (\sigma^i, \sigma^j) = \left\{ \begin{array}{ll}
1 & \text{if } i + j < n \\
\alpha & \text{if } i + j \geq n
\end{array} \right.$$

Check: $\delta_{\sigma, \alpha}$ is normalized 2-cocycle

Def: $A = (G, E_1F, \sigma)$ is the crossed product algebra wrt. $G$ and $\delta_{\sigma, \alpha}$, called cyclic algebra
\[ A = \bigoplus_{i=0}^{n-1} E e^i \]
\[ e \cdot b = \sigma(b) \cdot e, \quad b \in E \]
\[ e^n = a \]
Theorem (Brumer, Hasse, Noether)

Every cyclic group is admissible over $\mathbb{Q}$

Theorem (Schacher, 1968)

If $G$ is admissible over $\mathbb{Q}$ then all Sylow subgroups of $G$ are metacyclic ($\leq$ extension of cyclic by cyclic).

("Sylow-metacyclic")

Known for certain classes, e.g. solvable groups (Sonn)

Conjectured in general (Schacher)
Theorem (H&H, D. Krashen)

$K$ complete discretely valued field with alg. closed residue field $k$

$F$ a one variable function field over $K$

$G$ a finite group, $\text{char}(k) + 161$

Then:

$G$ admissible over $F$ $\iff$ all Sylow subgroups of $G$ are abelian metacyclic
Here: \( F = \frac{k(t)(x)}{K} \)

WTS: \( C \) admissible \( \Rightarrow \) every flow subgraph of \( C \) is oblique metacyclic.

Recall: \( A \cdot B \ csa \Rightarrow A \otimes B \ csa \)

Wedderburn: Every csa is of the form \( \text{Mat}_n(D) \) some division algebra \( D \).

Let define tensor product of division algebras

\( A \cdot B \) algebra, \( A \otimes B = \text{Mat}_n(D) \), define: \( A \cdot B = D \)
$Br(F) : \text{ set of division alg. } IF \text{ with two multiplicative Brauer group}$

d $\in Br(F)$

$\text{per}(\alpha) : \text{ order in } Br(F), \text{ always finite.}$

$F$ as before, $S$ set of discrete valuations on $F$

$\nu \in S$, let $k_{\nu}$ denote residue field

$Br(F)' := \{ \alpha \in Br(F) : (\text{per}(\alpha), \text{char}(k_{\nu})) = 1 \text{ all } \nu \in S \}$

Known: $\nu \in S \Rightarrow$ $\exists$ homomorphism

$\ln_{\nu} : Br(F)' \rightarrow H^2(k_{\nu}, O/2)$
Define \( \text{rank} : \text{Br}(F)' \to \prod_v H^2(k_v, O_{/2}) \) \( v \in S \).

Say that \( \alpha \in \text{Br}(F)' \) is determined by ramification if

\[ \text{per}(\alpha) = \text{per}(\text{ram}_v(\alpha)) \]

for some \( v \in S \).

Colliot - Thélèse, C. Jaguar, Ramíndel: Fix prime \( p \)

\[ \exists S \subseteq S \text{ s.t.} \]

1) \( \text{ram}_v \) is injective

2) none of the residue fields \( k_v \) (\( v \in S \))

has char. \( p \).
\( \delta \in \mathfrak{Br}(F) \Rightarrow \delta = \delta_p + (\delta - \delta_p) \)

s.t.
\[ \text{per}(\delta)_p = \text{per}(\delta_p) \]
\[ (\text{per}(\delta - \delta_p), p) = 1 \]

**Lemma:** \( \delta \in \mathfrak{Br}(F) \), \( \mathfrak{J} \) as above

= \( \delta_p \) is determined by ramification.

**Proof idea:**

1) Fix prime \( p \) arbitrarily (\( \mathfrak{J} \))
2) Let \( \mathfrak{I} \) as above
D: $\mathbb{G}$-crossed product division alg., max. subfield $\mathbb{E}$

$L|D_p$ is determined by ramification (lemma)

w.r.t. that $\mathbb{V}$

$\hat{E}/\hat{E}^P$ when $P$ is $p$-Sylow subgroup.