ANALYTIC NUMBER THY?

- How many pairs of coprime integers in $[1, N] \times [1, N]$?

- If $X$ a proj. variety, how many pts in $X(\mathbb{Q})$ of height at most $N$?
- How many primes \( \leq N \)?
- How many totally real cubic fields with discriminant \( \leq N^2 \)?
- How many totally real cubic fields with prime discriminant \( \leq N^2 \)?
- Autocorrelation of Möbius (HomA)

\[ \sum_{n \leq N} \mu(n) \mu(n+1) = o(N) \]

- What is the probability that a quadratic imaginary field \( \mathbb{Q}(\sqrt{-d}) \) (d random \([N, 2N])\) has class number prime to 7? (Cohn–Lenstra)
If \( n \) is a random squarefree in \( [N, 2N] \), what is the probability that \( E \) a totally real quintic field \( K/\mathbb{Q} \) with discriminant \( n \)?

\[ e^{-1/120} \]
"How many" is meant asymptotically

\[
\begin{align*}
1 & \times \frac{1}{4} \in [1, \sqrt{7}] \times \left[ \frac{1}{4}, \sqrt{7} \right] \\
\text{compare } 31 & = \frac{6}{\pi^2} N^2 \\
\text{NO}
\end{align*}
\]

\[
\lim_{N \to 000} N^{-2} \int \left( x/4 \right) \text{compare } 31 = \frac{6}{\pi^2}
\]
Or more:

\[ |(x+y) \text{ coprime in} \ \{1, N\} \times \{1, N\}| \]

\[ = \frac{6}{\pi^2} N^2 + O(N^{2-\delta}) \quad \text{for } \delta > 0 \] (power-saving error term)
Mostly we will consider just two fields: $\mathbb{Q}$ and $\mathbb{F}_q(t)$

$$\mathbb{Q}$$

$$= x \in \mathbb{Q} : \ |x|_p \leq 1 \text{ for all absolute values }$$

$$1 \text{ except } 1 \text{ for } p \neq \infty$$
Analogously:

\[ F_q[t] \subset F_q(t) \]

\[ = x \in F_q(t) \]

For each point \( P \) of \( IP' \),

\[ |x|_P = q \]

\[ ord_\infty(x) = \deg Q - \deg P \]
Analogous subring of $\mathbb{F}_q(t)$ is 
$$x : \|x\|_p \leq 1 \text{ all } P \text{ except } \infty$$

i.e. $x$ has no denominator

i.e. $x$ has no poles arbitrarily away from $\infty$

i.e. $x$ is a polynomial $P$

$$\|x\|_\infty = q^{\deg P}$$
A difference:

in $\mathbb{Q}$, $\infty$ is special
(there is only archimedean place)

in $\mathbb{F}_q(t)$, $\infty$ is not special

we can apply an automorphism of
$\mathbb{P}^1$ to move it around, e.g.

$\mathbb{F}_q[t, t^{-1}]$
possible integer-coset representations

monic polynomials coset representatives for

\[ \mathbb{F}_q[t]/(\mathbb{F}_q[t])^* \]
an interval in $\mathbb{Z}$ is
$n: \lfloor n \rfloor \leq d$

an interval in $\mathbb{F}_q[t]$ is
$f: |f-f_0| \leq e$

$\| \deg (f-f_0) $
e.s.

\[ f : 1 f - \mathbb{E} x \leq q^{n-1} \]

= monic polynomials of degree \( n \).
SQUAREFREE INTEGERS
& SQUAREFREE POLYNOMIALS

Q: How many integers in [N, 2N] are squarefree?

One might expect Prob (squarefree) to be

\[(1 - \frac{1}{5})(1 - \frac{1}{9})(1 - \frac{1}{25}) \ldots\]
and indeed this is so:

If \( sf(N) = \# \text{squares in } [N/2N] \),

then

\[
\lim_{N \to \infty} N^{-1} \, sf(N) = \prod_p \left(1 - p^{-2}\right) = \frac{\zeta(2)}{\pi^2}
\]
Over $\mathbb{F}_q[t]$, our interval is monic polynomials of degree $n$

\[ x^n + a_1 x^{n-1} + \ldots + a_n \]

an interval of size $q^n$

(So think of $q^n$ as $N$)
\[ s_{f_q}(n) = \text{# monic squarefree polys of degree } n \]

\[
\lim_{n \to \infty} q^{-n} s_{f_q}(n) = 1 - \frac{1}{q}
\]

Heuristically, one might expect

\[
\lim_{n \to \infty} q^{-n} s_{f_q}(n) = \prod_{\text{irreducible}} (1 - q^{-2 \deg P})
\]
\[ \prod_{p} (1 - p^{-2}) \]

\[ = \sum_{\mathfrak{p}} (2)^{-1} \]

\[ = \sum_{\mathfrak{p} \in \mathcal{O}} (1 - \frac{1}{q}) \]