Counting Abelian # Fields

Count $\mathbb{Z}/3\mathbb{Z}$ fields $\mathbb{E} \mathbb{N}_{\mathbb{Z}/3\mathbb{Z}} c_3(x)$

Need to count: $J_{\mathfrak{a}} \rightarrow \mathbb{Z}/3\mathbb{Z}$

restricted, so counting

$\prod_{\mathfrak{p}} \mathbb{Z}_\mathfrak{p}^* \rightarrow \mathbb{Z}/3\mathbb{Z}$
For each $p$, what are maps

$$\mathbb{Z}_p^* \to \mathbb{Z}_{3\mathbb{Z}}$$

$p \neq 3$ such maps factor

$$\mathbb{Z}_p^* \to \left( \mathbb{Z}_{p\mathbb{Z}} \right)^* \to \mathbb{Z}_{3\mathbb{Z}}$$

cyclic group order $p-1$

$p \equiv 1 \pmod{3}$

3 maps

$p \equiv 2 \pmod{3}$

1 map
2 non-trivial maps each have

\[ \text{Disc } p^2 \]

\[ f(s) = \sum_{n \geq 1} a_n n^{-s} = \prod_{\text{prime } p \equiv 1 \mod 3} \left(1 + 2p^{-2s}\right) x (1 + 2 \cdot 3^{-4s}) \]

\[ a_n = (\# \text{ cyclic cubic fields with } \text{disc } | = n) \]

\[ \times 2 \]
Let \( \chi_3 \), Dirichlet character mod 3

\[
\chi_3(1) = 1 \quad \chi_3(2) = -1
\]

\[
\sum (2s) \Lambda (2s, \chi_3)
\]

\[
= \prod_{\mathfrak{p}} \left( 1 + \mathfrak{p}^{-2s} + \cdots \right) \prod_{\mathfrak{p}} \left( 1 + \chi(\mathfrak{p})\mathfrak{p}^{-2s} + \cdots \right)
\]

\[
= \prod_{\mathfrak{p} \equiv 1 \mod 3} \left( 1 + 2\mathfrak{p}^{-2s} + \cdots \right) \prod_{\mathfrak{p} \equiv 2 \mod 3} \left( 1 + 0\mathfrak{p}^{-2s} + \cdots \right) \prod_{\mathfrak{p} \equiv 3 \mod 3} \left( \cdots \right)
\]
\[ f(s) \] \quad \text{abs. convergence} \quad \text{Re } s > \frac{1}{4} \\
\frac{\zeta(2s)L(2s,x_3)}{\zeta(2s)} \]

So \( f(s) \) has same rightmost pole behavior as \( \frac{\zeta(2s)L(2s,x_3)}{\zeta(2s)} \) \([s = \frac{1}{2}]\)

Applying Tauberian thm obtain

\[ N_{2/3Q}(x) \sim c x^{1/2} \quad \text{some constant } c. \]
Local Conditions

\[ N_{\frac{1}{2}a}, \Sigma(x) \quad \Sigma \text{ split comp at } 3 \]

\[ J_{\alpha} \to \mathbb{Z}_{\frac{1}{2}} \mathbb{Z} \quad \text{CFT split comp at } 3 \]

\text{iff } 1) \mathbb{Z}_3^* \to \mathbb{Z}_{\frac{1}{2}} \mathbb{Z} \text{ trivial}

2) \; 3e \in \mathbb{Z}_3^* \to 0

\( (1, 3, 1, 1, \ldots ) = (3^{-1}, 1, 3^{-1}, 3^{-1}, \ldots ) \)

\( J_{\alpha} \uparrow \text{ each element is in } \mathbb{Z}_p^* \)
Where does 3 go in $\mathbb{Z}_p^* \to \mathbb{Z}_{2^2}$ when $p \neq 3$?

3 is a $\mathbb{Z}_p^*$ if 3 $\not\equiv 0 \pmod{p}$

3 not $\mathbb{Z}_p^*$ if 3 $\equiv 1 \pmod{p}$

$\chi$ character: $\chi(p) = -1$ if 3 $\equiv 0 \pmod{p}$

$\chi(p) = 1$ if 3 $\equiv \pm 1 \pmod{p}$
\[ \frac{1}{2} \left( \prod_{P} \left( 1 + p^{-s} \right) + \prod_{P} \left( 1 + \chi(p)p^{-s} \right) \right) \]

ignore \( p=2,3 \)

counted \( J_0 \to \frac{3}{2} \mathbb{Z} \)

counts \( J_0 \to \mathbb{Z}/2\mathbb{Z} \)

w/ sign + if \( (3,1,3,3,\ldots) \mapsto 0 \)

sign - \( (3,1,3,3,\ldots) \mapsto 1 \)

counts quad fields s.c. @ 3
So apply Tauberian theorem

first Euler product gives rightmost pole $s = 1$

can ignore second Euler product

bc it is analytic past on $\text{Re } s \geq 1$

In general, any abelian $G$, any local conditions $\Sigma$ (finitely many places)
Write
\[ f(s) = \sum \text{Euler products} \]

\[ \uparrow \]

Dirichlet series

\[ N_{6, \Sigma} \]

\[ \uparrow \]

identify rightmost poles all Euler products

For example, independence at different primes fails unless
\[ G = \mathbb{Z}/p^2 \mathbb{Z} \]
Grunwald–Wang

There are local extensions that never happen from a global extension.

There is no $\mathbb{Q}_2$ extension $K$ of $\mathbb{Q}_2$ for which $K_2$ is an unramified extension of $\mathbb{Q}_2$ of degree 8 (equivalent in which 2 is totally inert).
Over $K$ number field, also can have local extns at diff places that can each occur from global extn, but can’t occur together.

Bad reasons can be eliminated if instead of counting by discriminant, we count by conductor.

\[ \text{class field theory} \]

\[ \text{product of ram primes to some powers} \]
When counting by conductor, you have nice behavior, independence except for when 6-W interfere.

So what should we expect when counting general number fields? (Open question: What should we be counting them by?)
Heuristic

Count $G_{a,n} \to G < S_n$

(If $G$ abelian, $G_{a,n} \to G$ built out of maps $G_{a,p} \to G$ in some way)

$\lambda$ asym. behaves as if $G_{a,n} \to G$ (nonabelian)

built out of $G_{a,p} \to G$ in a similar way