To count cubic rings we need to count $GL_2(\mathbb{Z})$ classes of $ax^3 + bx^2y + cx y^2 + dy^3$

$f = an x^n + a_{n-1} x^{n-1} y + \ldots + a_0 y^n \quad a_i \in \mathbb{Z}$

$\text{Spec } \mathbb{Z}$

$\text{Spec } \mathbb{R}_f$

Global functions on $V_f$

$H^0(V_f, \mathcal{O})$ rank n ring $\mathbb{R}_f$
\((a_1, \ldots, a_n) = 1\)

\[ \text{Spec } \mathbb{Z} \]

\[ V_f = \text{Spec } R_f \]

\[ \text{Spec } R \]

\[ \text{Spec } R_f \]

\[ \text{Spec } \mathbb{Z} \]

rank n ring (e.g. maximal order in degree n field)
Spec $\mathbb{R}_f$ \implies \text{has a canonical embedding CPN}

What is a map $\text{Spec } \mathbb{R}_f \rightarrow \text{CPN}$?

- line bundle on Spec $\mathbb{R}_f$
- $N+1$ sections of bundle, nowhere all vanishing
- element of class group of $\mathbb{R}_f$
- elements of the ideal
Canonical embedding of Spec $R_f$

given ideal class of inverse different

How many sections do you need so they are nowhere all vanishing?

In general need $n-1$.

cubic ring Spec $R_f \subset \mathbb{P}^1_\mathbb{Z}$
Rmk. For every \( n \), some \( \text{Spec } R_f \) do embed in \( \mathbb{P}^1_{\mathbb{Z}} \). These are particularly nice rank \( n \) rings (orders in \( \text{deg } n \) fields).

Analogy to plane curves.

Thm. \( \text{GL}_2(\mathbb{Z}) \times \text{GL}_n(\mathbb{Z}) \times \text{GL}_n(\mathbb{Z}) \) orbit of \( \mathbb{Z}^2 \times \mathbb{Z}^n \times \mathbb{Z}^n \) parametrize class groups of these \( \text{Spec } R_f \subset \mathbb{P}^1_{\mathbb{Z}} \).

\( n=3 \) all cubic rings are this nice.
How to count \( \text{GL}_2(\mathbb{Z}) \) orbits of \((c,b,c,d)\)?

\[ \mathbb{Z}^4 \] 4 dim lattice of \((a,b,c,d)\)

Need a fundamental domain \( F \) for action of \( \text{GL}_2(\mathbb{Z}) \)

each orbit has exactly 1 lattice point in \( F \)
Count lattice points in $F$

$|\text{Disc}| < X$

Geometry of numbers

- First example: region scaling with $X$
  $B$ ball
  $\#\text{pts in } B \cdot X = \text{vol}(B) X^{\dim}$
region $R$ + I want to count lattice points in $R$ ($N(R) \approx \# \text{lattice pts in } R$)

Hope:
\[ N(R) \approx \text{Vol}(R) \]

How big can $N(R) - \text{Vol}(R)$ be?

\[ \text{Vol} = A \]

could have as many as $2A$ points as few as zero
Thm (Davenport)

\[ |N(R) - \text{Vol}(R)| = O\left(\text{Vol}(\text{Proj}(R))\right) \]

\[ \text{Vol}(\text{Proj}(R)) \]

projections of \( R \) onto all coordinate hyperplanes (of every lower dim)

\[ \text{Vol} = A^2 \]

\[ \text{Vol}(\text{Proj}) = A \]
\[
\frac{1}{A}, \frac{1}{A} \quad A^2
\]

\[N(R) \text{ could be } A^2 \text{ or } 0\]

Vol 1

2dime Proj

Vols: \(A, \frac{1}{A^2}\)

1dime Proj

Vol: \(A^2\)
\[ |\text{Disc}| \leq X \]

\[ \text{Vol} \left( F \cap \{|\text{Disc}| \leq X\} \right) \sim c X \]

Largest Vol of a Proj = \( \infty \)
Points here, e.g. with $a=0$

Determine we didn't want these points anyway.

This is an "order" in $\mathbb{Q} \oplus K \leftrightarrow$ quadratic field
a lot of lattice pts w/ Disc=0

e.g. corr to rings like \( \mathbb{Z}[x]/x^3 \)

Now we make a smaller region

\( |\text{Disc}| \geq 1 \)

\( |a| \geq X^{16} \)
D-H: \( N_{S_3}(X) \sim \frac{1}{3 \bar{s}(3)} X \)

Many other similar parametrizations of algebraic objects are now known.

One can apply geom of #s ideas to count the objects.

Improvements to geom of #s, due to Bhargava's work on counting quartic #fields.
Finding a fundamental domain

Use a fundamental domain

\[ F = \GL_2(\mathbb{R}) \backslash \GL_2(\mathbb{Z}) \]

\( v_0 \), some binary cubic form

\( F_{v_0} \) fund domain for \( \GL_2(\mathbb{Z}) \) on \( (a, b, c, d) \)
You can choose different \( v_0 \), take a lot of \( v_0 \)s and get a lot fundamental domains.

Averaging over these, improves error.