I) Hodge decomposition

$X/\mathbb{C}$ smooth proj

Thm: $\exists$ a natural isom.

$$H^n(X^{an}, \mathbb{Q}) \otimes \mathbb{C} \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{C}}).$$

ex: $X = E$ elliptic curve $/\mathbb{C}$ $\rightsquigarrow$ $E = \mathbb{C}/\Lambda$

Thm $\Rightarrow$ $H^0(X, \Omega^1_X) \hookrightarrow H^1(X^{an}, \mathbb{Q}) \otimes \mathbb{C}$

$H^0(X, \Omega^1_X) = \mathbb{C} \cdot \omega$, $H^1(X^{an}, \mathbb{Q}) \otimes \mathbb{C} = \mathrm{Hom}(\Lambda, \mathbb{C})$.

$$\omega \longmapsto \left(\gamma \in \Lambda \mapsto \int_\gamma \omega\right).$$

HIGHLY TRANSCENDENTAL

Cor: Say $f : X \longrightarrow Y$ of smooth proj. vars

& $f^* : H^n(Y, \mathbb{Q}) \xrightarrow{\sim} H^n(X, \mathbb{Q})$

$\Rightarrow$ $H^i(X, \Omega^j_X) \xleftarrow{\sim} H^i(Y, \Omega^j_Y) : f^*$

$\forall\, i+j=n$

II) Etale cohomology:

Say X is a scheme,

$A \in \{\mathbb{Z}/n, \mathbb{Z}_p, \mathbb{Q}_p\}$ (p prime)

$\xRightarrow{\text{Grothendieck}}$ $H^*(X_{et}, A)$ algebraically defined

Thm (Artin): $X/\mathbb{C}$ variety

$\Longrightarrow$ $H^*(X_{et}, A) \simeq H^*(X^{an}, A)$.

Upshot: Say X is defined $/\mathbb{Q}$.

$\xRightarrow{\text{Thm} + \varepsilon}$ $\exists$ a natural action

$G_{\mathbb{Q}} \curvearrowright H^*(X^{an}, A)$

<u>ex</u>:

i) $X = E$ elliptic curve $/\mathbb{C}$, but defined $/\mathbb{Q}$.

$\therefore \quad E \simeq \mathbb{C}/\Lambda$

$$H^1(X^{an}, \mathbb{Z}/n) = \mathrm{Hom}(H_1(X^{an}, \mathbb{Z}/n), \mathbb{Z}/n)$$
$$\cong \mathrm{Hom}(\Lambda, \mathbb{Z}/n)$$
$$\cong E[n]^\vee$$

Thm $\Rightarrow$ $E[n]$ is defined $/\overline{\mathbb{Q}}$

$\therefore$ get $G_{\mathbb{Q}} \circlearrowright E[n]$

Set $T_pE = \varprojlim E[p^n]$

$\therefore$ get a conts. $G_{\mathbb{Q}}$-action on T_pE $\longleftrightarrow$ dual to the $G_{\mathbb{Q}}$-action on $H^1(X^{an}, \mathbb{Z}_p)$.

2) $X = \mathbb{G}_m$

Some analysis shows

$$H^1(\mathbb{G}_m^{an}, \mathbb{Z}/n) \cong \mu_n^\vee$$

Set $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$

$\therefore$ $G_{\mathbb{Q}}$-action on $\mathbb{Z}_p(1)$ $\overset{\text{dual}}{\longleftrightarrow}$ $G_{\mathbb{Q}}$-action on $H^1(X^{an}, \mathbb{Z}_p)$

<u>Notation</u>: For any $\mathbb{Z}_p$-algebra R,

set $R(i) := R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes i}$

<u>Note</u>: if $G_{\mathbb{Q}} \curvearrowright R$, it also acts on $R(i)$

3) $X = \mathbb{P}^1$

$$H^2(\mathbb{P}^{1\,an}, \mathbb{Q}_p) \underset{\substack{\uparrow \\ \text{as } G_{\mathbb{Q}}\text{-modules}}}{\cong} H^1(\mathbb{G}_m^{an}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-1)$$

More generally, if X smooth proj of dim d, then

$$H^{2d}(X^{an}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-d).$$

III) <u>Hodge - Tate decomposition</u>

Fix a prime p, $K/\mathbb{Q}_p$ finite ext,

$$K \subset \underset{\substack{\circlearrowleft \\ G_K = \mathrm{Gal}(\overline{K}/K)}}{\overline{K}} \subset \widehat{\overline{K}} = \mathbb{C}_p \circlearrowleft G_K$$

Thm (Hodge-Tate decomp)

Say $X|K$ smooth proj variety.

$\Rightarrow \exists$ a natural G_K-equivariant isom

$$H^n(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X|K}) \otimes_K \mathbb{C}_p(-j)$$

where G_K acts in the natural way on both sides.

To use this theorem, use:

Thm (Tate). Fix $i \neq j \in \mathbb{Z}$

$$\mathrm{Hom}_{G_K}(\mathbb{C}_p(i), \mathbb{C}_p(j)) = 0$$

$$\mathrm{Ext}^1_{G_K}(\mathbb{C}_p(i), \mathbb{C}_p(j)) = 0$$

ex:

1) $X = \mathbb{P}^1 / K$, $n=2$.

$$H^2(X_{\bar{K}}, \mathbb{Q}_p)\otimes\mathbb{C}_p \cong \big(H^2(X,\mathcal{O}_X)\otimes\mathbb{C}_p\big)\oplus\big(H^1(X,\Omega^1_X)\otimes\mathbb{C}_p(-1)\big)\oplus \big(H^0(X,\Omega^2_X)\otimes \mathbb{C}_p(\cdots$$

$$H^2(X_{\bar{K}}, \mathbb{Q}_p)\otimes\mathbb{C}_p = \mathbb{Q}_p(-1)\otimes\mathbb{C}_p = \mathbb{C}_p(-1).$$

$$H^2(X,\mathcal{O}_X)\otimes\mathbb{C}_p = 0, \qquad H^1(X,\Omega^1_X)\otimes\mathbb{C}_p(-1) = \mathbb{C}_p(-1), \qquad H^0(X,\Omega^2_X)\otimes\mathbb{C}_p(\cdots) = 0$$

2) $X = E$ ell. curve $/K$.

$$H^1(X_{\bar{K}}, \mathbb{Q}_p)\otimes\mathbb{C}_p = \big(H^1(X,\mathcal{O}_X)\otimes_K\mathbb{C}_p\big)\oplus\big(H^0(X,\Omega^1_X)\otimes_K\mathbb{C}_p(-1)\big)$$

$$H^1(X_{\bar{K}}, \mathbb{Q}_p)\otimes\mathbb{C}_p \cong T_p(E)^\vee\otimes\mathbb{C}_p.$$

$$H^1(X,\mathcal{O}_X)\otimes_K\mathbb{C}_p = \mathbb{C}_p = \mathrm{Lie}(E^\vee)\otimes\mathbb{C}_p$$

$$H^0(X,\Omega^1_X)\otimes_K\mathbb{C}_p(-1) = \mathbb{C}_p(-1) = \mathrm{Lie}(E)^\vee\otimes\mathbb{C}_p(-1)$$

~~Set~~

Cor: X/K smooth proj.

$$\Rightarrow H^i(X, \Omega^j_{X/K}) \cong \left(H^{i+j}(X_{\bar{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p(j) \right)^{G_K}$$

Rmk: Ito used this Cor. to reprove:

Thm: X, Y Calabi-Yau varieties $/ \mathbb{C}$

$X \underset{bir}{\sim} Y$.

$$\Rightarrow h^{i,j}(X) = h^{i,j}(Y)$$

$$h^{i,j}(X) = \dim H^j(X, \Omega^i_{X/K})$$

Rmk: $\exists$ a good variant for general X

IV) Hodge-Tate Spectral Sequence

Use perfectoid spaces to prove:

Thm (HT ss):

$C/\mathbb{Q}_p$ complete & algebraically closed

X/C proper smooth rigid-analytic space

$\Rightarrow \exists$ an E_2-spectral sequence

$$E_2^{ij} : H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}(X, \mathbb{Q}_p) \otimes C$$

$\rightsquigarrow$ get Hodge-Tate filtration on $H^n(X, \mathbb{Q}_p) \otimes C$

Rmk:

1) HT ss is functorial

$\Rightarrow$ If X is defined $/K$ (with $K/\mathbb{Q}_p$ finite)

then Tate's thm

$\Rightarrow$ get HT decomposition for X.

2) The HT ss always degenerates (Conrad-Gabber)

but not canonically so:

ex: Say $X = E$ ell. curve

HT ss $\Rightarrow$ low degree SES

$$0 \to H^1(X, \mathcal{O}_X) \xrightarrow{\;\overset{s}{\leftarrow}\;} H^1(X, \mathbb{Q}_p) \otimes \mathbb{C}_p \to H^0(X, \Omega^1_X)(-1) \to 0$$

- ~~maps~~ maps go the wrong way
- Cannot choose a splitting that varies well in family