①

Integral aspects of Hodge-Tate "decomp"

Say $C/\mathbb{Q}_p$ alg closed + complete

X/C smooth proper

Yesterday: $\dim_{\mathbb{Q}_p} H^n(X, \mathbb{Q}_p) = \sum_{i+j=n} \dim H^i(X, \Omega^j_{X/C})$

Q: What about integral / mod-p analogs?

Assume $\mathfrak{X}/\mathcal{O}_C$ proper smooth formal scheme, $\mathfrak{X}_C = X$

Let $k = \mathcal{O}_C/\mathfrak{m}$ residue field.

Thm: i) We always have

$$\dim_{\mathbb{F}_p} H^n(X, \mathbb{F}_p) \leq \cancel{\sum_{i+j=n} \dim_k \left(H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k}\right)}$$

$$\sum_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k})$$

2) The inequality can be strict.

Rmk: Previous work by Caruso - Faltings.

II) Fontaines A_{inf}

Def:
$$\begin{tikzcd} A_{inf} := W(\mathcal{O}_C^\flat) \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_C \arrow[d] \\ \mathcal{O}_C^\flat := \varprojlim_{\phi} \mathcal{O}_C/p \arrow[r] & \mathcal{O}_C/p \end{tikzcd}$$

Choose $\underline{p} = (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathcal{O}_C^\flat$

$\rightsquigarrow [\underline{p}] \in A_{inf}$

Philosophy: " $A_{inf} = \mathcal{O}_C^\flat \hat{\otimes}_{\mathbb{F}_p} \mathbb{Z}_p$ "

$\therefore$ A_{inf} should behave like a regular local ring of dim 2 with co-ordinates $[\underline{p}]$ & p.

$\therefore$ Have maps

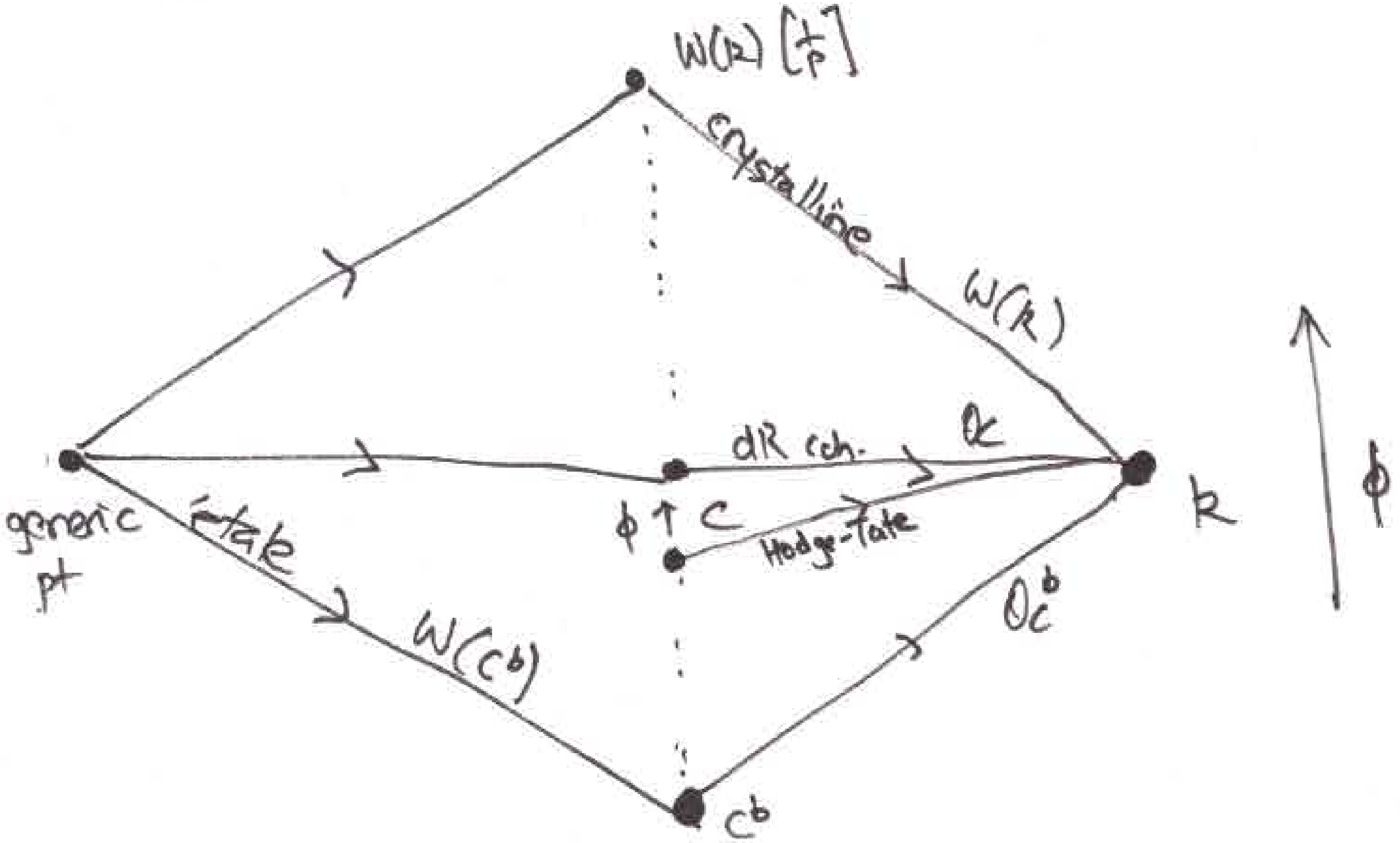
$A_{inf} \longrightarrow \mathcal{O}_C^b$ kill p

$A_{inf} \xrightarrow{\theta} \mathcal{O}_C$ kill $p - [\underline{p}]$

$A_{inf} \longrightarrow W(k)$ kill $[\underline{p}]$ (*)

$A_{inf} \longrightarrow W(C^b)$ invert $[\underline{p}]$

Spec (A_{inf})


W(k)[1/p]
crystalline
W(k)
dR ch.
k
generic pt
étale
φ ↑ C
Hodge-Tate
W(C♭)
O_{C♭}
C♭
k
φ

III) More precise result

$\mathfrak{X}/\mathcal{O}_C$ proper smooth

Thm: $\exists$ a naturally attached perfect complex $R\Gamma_A(\mathfrak{X})$ of A_{inf}-modules with the foll. comparison isoms:

a) Étale cohomology:

$$R\Gamma_A(\mathfrak{X}) \otimes W(C^\flat) \simeq R\Gamma(X, \mathbb{Z}_p) \otimes W(C^\flat)$$

b). Hodge-Tate cohomology:

Set $\tilde{\theta} = \theta \circ \phi^{-1}$

$\exists$ an E_2 spectral seq.

$$E_2^{ij}: H^i(\mathfrak{X}, \Omega^j_{\mathfrak{X}/\mathcal{O}_C}) \Longrightarrow H^{i+j}(\tilde{\theta}^* R\Gamma_A(\mathfrak{X}))$$

c) de Rham cohomology:

$$\theta^* R\Gamma_A(\mathfrak{X}) \simeq R\Gamma_{dR}(\mathfrak{X}/\mathcal{O}_C).$$

How to get numerical consequences?

<u>Claim</u>: $\dim_{\mathbb{F}_p} H^n(X, \mathbb{F}_p) \leq \sum_{i+j=n} \dim H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k})$

<u>Pf</u>: Consider the perfect complex

$$M = R\Gamma_A(\mathfrak{X})$$

$$\begin{array}{ccc}
\dim_{C^b} H^n(M \otimes C^b) & \leq & \dim_k H^n(M \otimes k) \\
\| \text{ by (1)} & & \| \text{ by (3)} \\
\dim_{\mathbb{F}_p} H^n(X_{et}, \mathbb{F}_p) & & \dim_k H^n_{dR}(\mathfrak{X}_k/k) \\
 & & \wedge| \\
 & & \sum_{i+j} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k})
\end{array}$$

Strategy: define a complex $A\Omega_{\mathfrak{X}}$ of A_{inf}-modules on $\mathfrak{X}$ and

Set $R\Gamma_A(\mathfrak{X}) := R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}})$

IV) A first pass:

Consider the map

$$\nu : X_{\mathrm{pro\acute{e}t}} \longrightarrow \mathfrak{X}$$

Fontaine's construction gives a sheaf

$$A_{\mathrm{inf},X} := A_{\mathrm{inf}}(\hat{\mathcal{O}}_X^+)$$

on $X_{\mathrm{pro\acute{e}t}}$.

Primitive comparison thm gives:

Thm :

$$H^i(X_{\text{proét}}, A_{\text{inf},X}) \overset{a}{\cong} H^i(X, \mathbb{Z}_p) \otimes A_{\text{inf}}$$

Naive guess: setting

$$R\Gamma_A(\mathfrak{X}) := R\Gamma(\mathfrak{X}, R\nu_* A_{\text{inf},X})$$

does the job.

If this works, we would have

$$\tilde{\Theta}^* R\Gamma_A(\mathfrak{X}) := R\Gamma(\mathfrak{X}, R\nu_* \hat{\mathcal{O}}_X^+)$$

$\therefore$ would know that there is an

<u>integral</u> Hodge-Tate spec. seq. calculating

Yesterday's calculation:

Say $\mathcal{X} = \mathrm{Spf}\left(\mathcal{O}_C\langle T^{\pm 1}\rangle\right)$

$$\Longrightarrow H^i(\mathcal{X}, R\nu_* \hat{\mathcal{O}}_X^+)$$

$$\doteq H^i_{cts}\left(\mathbb{Z}_p(1), \mathcal{O}_C\langle T^{\pm \frac{1}{p^\infty}}\rangle\right)$$

$$= \widehat{\bigoplus_{j\in\mathbb{Z}}} H^i\left(\mathbb{Z}_p(1), \mathcal{O}_C \cdot T^j\right) \oplus \widehat{\bigoplus_{j\in\mathbb{Z}[\frac{1}{p}]-\mathbb{Z}}} H^i\left(\mathbb{Z}_p(1), \mathcal{O}_C \cdot T^j\right)$$

$$\widehat{\bigoplus_{i\in\mathbb{Z}}} H^i\left(\mathcal{O}_C \cdot T^j \xrightarrow{0} \mathcal{O}_C \cdot T^j\right) \oplus \widehat{\bigoplus_{j\in\mathbb{Z}[\frac{1}{p}]\cdot\mathbb{Z}}}$$

$$\underbrace{H^i\left(\mathcal{O}_C \cdot T^j \xrightarrow{\varepsilon^j - 1} \mathcal{O}_C \cdot T^j\right)}$$

a lot of extra torsion!

V) $L\eta$ - construction

Say R is a ring, $f \in R$ nonzerodivis

Def: Given a chain complex $K^\cdot$ of f-torsionfree modules, define

$$(\eta_f K^\cdot)^i = \{ \alpha \in f^i K^i \mid d(\alpha) \in f^{i+1} K^{i+1} \}$$

$\therefore$ get a subcomplex $\eta_f K^\cdot \subseteq K^\cdot\left[\frac{1}{f}\right]$

Miracle: $H^i(\eta_f K^\cdot) = H^i(K^\cdot) / H^i(K^\cdot)[f]$

$\therefore$ η_f passes to derived category

$$L\eta_f : D(R) \longrightarrow D(R)$$

$$K^\cdot \longmapsto \eta_f K^\cdot$$

__Ex__: $R = \mathbb{Z}_p$, $f = p$

$$K = \mathbb{Z}/p$$

$$L\eta_f(K) = \eta_f\left(\mathbb{Z}_p \xrightarrow{p} \overset{\deg 0}{\mathbb{Z}_p}\right)$$

$$= \left(\tfrac{1}{p}\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p\right)$$

$$\simeq 0$$

$$L\eta_f(\mathbb{Z}/p^2) = \mathbb{Z}/p.$$

VI) Definition of $A\Omega_{\mathfrak{X}}$:

$\mathfrak{X}/\mathcal{O}_C$ smooth formal scheme

__Def__: $$A\Omega_{\mathfrak{X}} = L\eta_{[\varepsilon]-1}\left(R\nu_* A_{\inf,X}\right)$$