Goal: $\exists$ perfectoid space

$$\mathcal{X}^*_{\Gamma(p^\infty)} \text{ s.t.}$$

$$\begin{tikzcd} \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)} \sim \mathcal{X}^*_{\Gamma(p^\infty)} \arrow[d] \arrow[dr, "\pi_{HT}"] & \\ \mathcal{X}^*_{\Gamma} & \mathbb{P}^{1,ad} \end{tikzcd}$$

Recall: $\to \mathfrak{X}_n \to \dots \to \mathfrak{X}_1 \to \mathfrak{X}_0$

- tower of flat formal schemes $/ \mathrm{Spf}\, \mathbb{Z}_p^{cycl}$
- transition maps are relative Frobenius mod p

$\mathfrak{X}_\infty := \varprojlim_n \mathfrak{X}_n$, $\underset{\uparrow \text{ perfectoid space}}{\mathcal{X}_\infty} = (\mathfrak{X}_\infty)^{ad}_\eta$

Part of the tower

$$\left(X^*_{\Gamma_0(p^m)}\right)_m$$

has this structure

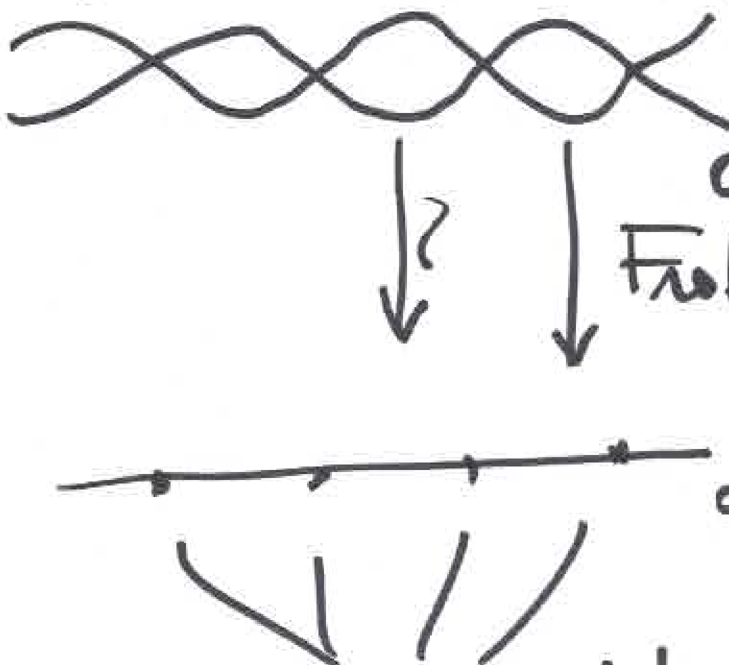
$$\Gamma_0(p^m) = \left\{ \gamma \in \Gamma \;\middle|\; \begin{array}{l} \gamma_p \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p^m \\ \det \gamma_p \equiv 1 \bmod p^m \end{array} \right\}$$

Consider: $\overline{X}^*_{\Gamma}$ = special fiber of ~~the~~ integral model over Spec $\mathbb{Z}_p$ of X^*_{Γ}

$\overline{X}^*_{\Gamma_0(p)}$ = — " — of $X^*_{\Gamma_0(p)}$

~~setting~~

/ $\mathbb{F}_p$



E_0, Γ-level structure

$D_0 \subset E_0[p]$ subgp scheme of order p

E_0, Γ-level structure

correspond to supersingular elliptic curves / $\overline{\mathbb{F}}_p$

$E_0 / \overline{\mathbb{F}}_p$ elliptic curve

$$\dim_{\overline{\mathbb{F}}_p} \left(E_0[p](\overline{\mathbb{F}}_p) \right) \leq 1$$

because $F : E_0 \longrightarrow E_0^{(p)}$

Frobenius isogeny

$\ker F \subset E_0[p]$

connected gp scheme

$$\begin{tikzcd}
\overline{X}^*_\Gamma \arrow[r, "\sim"] \arrow[dr, bend right, "\text{relative Frobenius}"'] & \overline{X}^*_{\Gamma_0(p)\infty} \arrow[d] \\
& \overline{X}^*_\Gamma
\end{tikzcd}$$

$E_0' \longmapsto (E_0'/\ker F, \ \ldots, \ E_0'[p]/\ker F)$

$(E_0, \ \Gamma\text{-level structure}, \ D_0 \subset E_0[p], \ D_0 \cap \ker F = \{0\})$

relative Frobenius

$(E_0'/\ker F, \ldots)$

Want to lift this to char 0:

$E/\mathbb{Z}_p^{cycl}$ ell curve w red E_0

$$0 \to \ker F \to E_0[p] \to G_0 \to$$

$\Rightarrow G$ lifts to char 0 (fin étale)

- get short exact sequence

$$0 \to C \to E[p] \to \underset{\text{lift of } \mathfrak{G}_0}{\mathfrak{G}} \to 0$$

$C = C_1$ canonical subgp of E of level 1.

- can consider formal scheme $/\mathbb{Z}_p^{cycl}$

$$\mathfrak{X}^*_\Gamma(0) = \mathfrak{X}^{*,\mathrm{ord}}_\Gamma$$

= p-adic completion of ordinary locus $/\mathbb{Z}_p^{cycl}$ along special fiber.

In our tower of formal schemes

$$\mathfrak{X}_n = \mathfrak{X}^*_\Gamma(0)$$

transition maps: canonical lifts of relative Frobenius

$$\dim_{\mathbb{F}_p}\left(E_0[p](\overline{\mathbb{F}}_p)\right) = \begin{cases} 1 & E_0 \text{ ordin} \\ 0 & E_0 \text{ supersi} \end{cases}$$

Can ~~see~~ see that E_0 is ordinary by checking whether $Ha(E_0/\overline{\mathbb{F}}_p)$ is invertible.

univ elliptic curve

$\mathcal{E}_0/\overline{X}_\Gamma \qquad V : \mathcal{E}_0^{(p)} \longrightarrow \mathcal{E}_0$

Verschiebung isogeny

Map induced by V on differentials determines

$$Ha(\mathcal{E}_0/\overline{X}_\Gamma) \in \omega^{\otimes(p-1)}_{\mathcal{E}_0/\overline{X}_\Gamma}$$

<u>RK</u>: Multiplication by $Ha(\mathcal{E}_0/\overline{X}_\Gamma)$ is equivariant for Hecke operators away from p.

- can also define
  - $\mathcal{X}^*_{\Gamma}(0) \subset \mathcal{X}^*_{\Gamma}$

    cut out by $|\mathrm{Ha}| = 1$.
  - $\mathcal{X}^*_{\Gamma_0(p)}(0)_{\mathrm{anti}} \subset \mathcal{X}^*_{\Gamma_0(p)}$

    = anticanonical part of ordinary locus at level $\Gamma_0(p)$

    = parametrizes (E, Γ-level structure, $D \subset E[p]$ s.t. $D \cap C_1 = \{0\}$)

Can lift diagram over $\overline{\mathbb{F}}_p$ to char 0:

$$\begin{tikzcd}
\mathcal{X}^*_{\Gamma}(\mathcal{O}) \arrow[r, "\sim"] \arrow[dr] & \mathcal{X}^*_{\Gamma_0(p)}(\mathcal{O})_{\mathrm{anti}} \arrow[d] \\
 & \mathcal{X}^*_{\Gamma}(\mathcal{O})
\end{tikzcd}$$

ordinary locus at level Γ

canonical lift of relative Frobenius

$$\begin{tikzcd}
E' \arrow[r, mapsto] \arrow[dr] & \left(E'/C_1', \; E'[p]/C_1'\right) \arrow[d] \\
 & E'/C_1'
\end{tikzcd}$$

mod p: quotienting by $\ker F$

<u>Upshot</u>: $\exists$ perfectoid space

$$\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_{\text{anti}}$$

ordinary locus

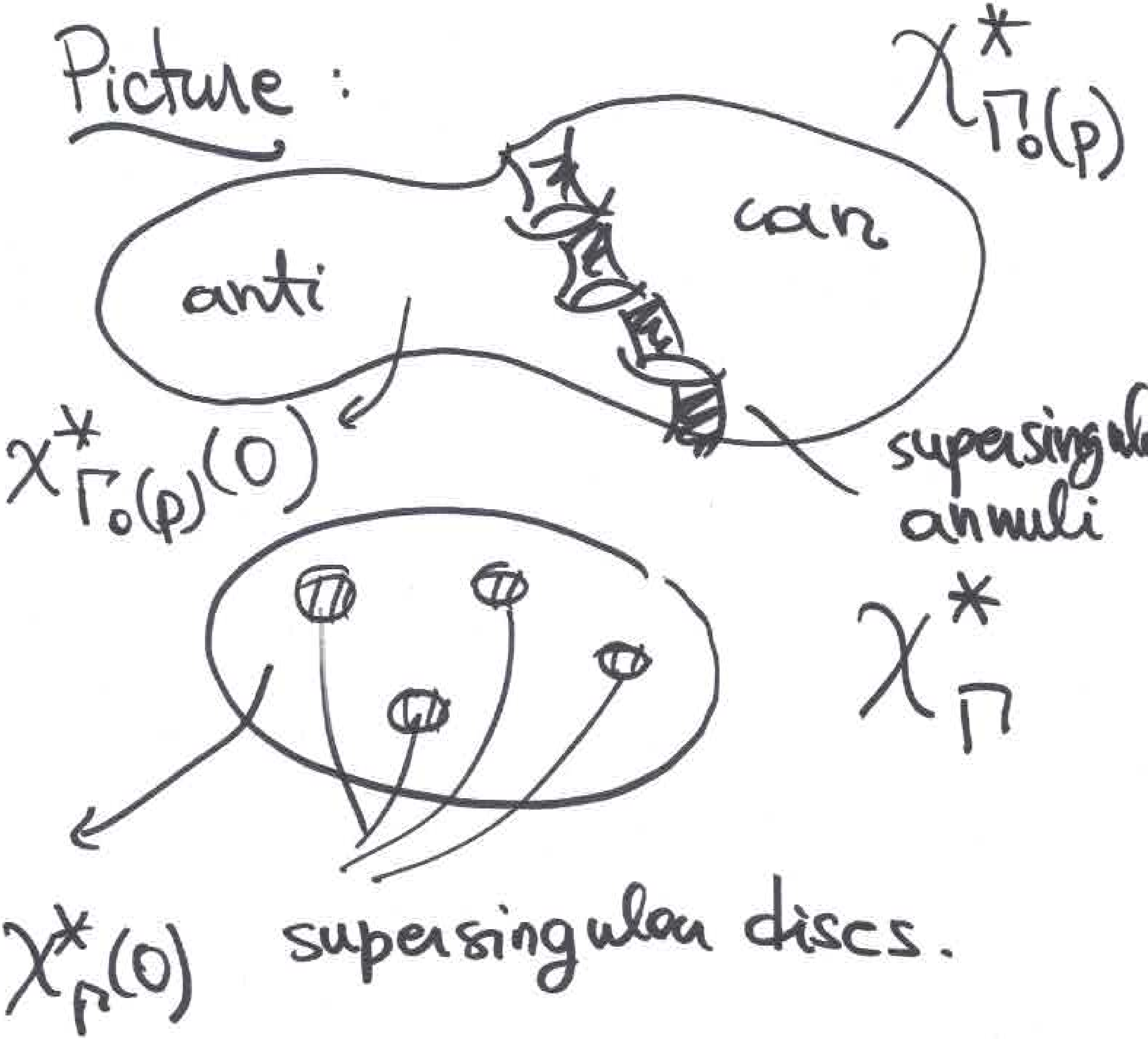
$D \subset E[p^\infty]$.
$D[p] \cap C_1 = \{0\}$.

s.t.

$$\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_{\text{anti}}$$

$$\sim \varprojlim_m \mathcal{X}^*_{\Gamma_0(p^m)}(0)_{\text{anti}}$$

Picture:


$X^*_{\Gamma_0(p)}$
can
anti
$X^*_{\Gamma_0(p)}(0)$
annuli
X^*_{Γ}
$X^*_{\Gamma}(0)$
supersingular discs.

Next steps:

1). For any $\varepsilon \in [0, 1/2)$ can define ε-nbhd of $X^*_\Gamma(0)$.

$X^*_\Gamma(\varepsilon) \subset X^*_\Gamma$

cut out by $|Ha| \geqslant p^\varepsilon$

any elliptic curve satisfying $|Ha| \geqslant p^\varepsilon$ will have a canonical subgp of level 1 C_1 lift of ker F mod $p^{1-\varepsilon}$

$$G_0 = E_0[p]/\ker F$$

"étale mod $p^{1-\varepsilon}$"

get diagram:

$$\begin{tikzcd}
\mathcal{X}^*_{\Gamma}(p^{-m}\varepsilon) \arrow[r, "\sim"] \arrow[dr] & \mathcal{X}^*_{\Gamma_0(p^m)}(\varepsilon)_{\mathrm{anti}} \arrow[d] \\
 & \mathcal{X}^*_{\Gamma}(\varepsilon)
\end{tikzcd}$$

canonical lift of relative Frobenius mod $p^{1-\varepsilon}$

<u>Upshot</u>: extend perfectoid structure to $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$

$\mathcal{X}^*_{\Gamma_0(p^\infty)}(\varepsilon)_{\text{anti}}$ is a strict nbhd of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_{\text{anti}}$ in $|\mathcal{X}^*_{\Gamma_0(p^\infty)}|$.

2). Go from level $\Gamma_0(p^\infty)$ to level $\Gamma(p^\infty)$.

$$\begin{array}{c} \mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\text{anti}} \\ \Big\downarrow \longleftarrow \text{finite étale} \\ \mathcal{X}^*_{\Gamma_0(p^m)}(\varepsilon)_{\text{anti}} \end{array}$$

Use almost purity

to show $\mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\text{anti}}$

is perfectoid.

Note: for modular curves,

maps are finite étale

at the boundary.

for Siegel modular varieties

$GSp_{2g},\ g > 1$

maps

$$\mathcal{X}^*_{\Gamma_1(p^m)} \to \mathcal{X}^*_{\Gamma(p^m)}$$

ramification

at boundary.

3). Howe

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\text{anti}}$$

· <u>affinoid</u> perfectoid

$$|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\text{anti}}|$$

$$\cap$$

$$|\mathcal{X}^*_{\Gamma(p^\infty)}|$$

$$\parallel$$

$$\varprojlim_m |\mathcal{X}^*_{\Gamma(p^m)}|$$

Finally, have

$GL_2(\mathbb{Q}_p) \circlearrowleft |\mathcal{X}^*_{\Gamma(p^\infty)}|$

use this action to translate perfectoid structure from

$\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\text{anti}}$

to the whole space.

Uses: Hodge-Tate period morphism.