1) The Hodge-Tate period map (+ applications) ①

§1. Recall:

1). $\exists$ perfectoid space over $\mathbb{Q}_p^{cycl}$:

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{anti} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{anti}$$

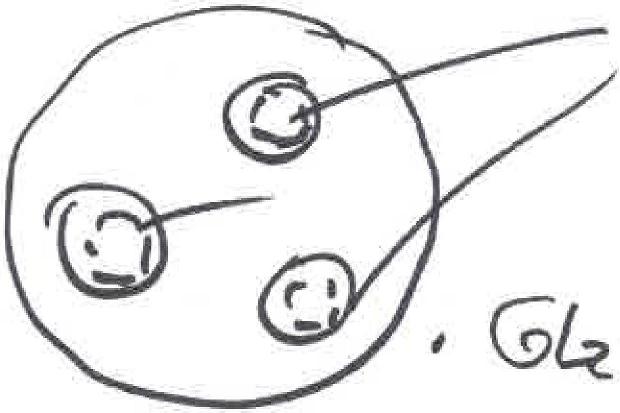
· $\varepsilon$-nbhd of anticanonical part of ordinary locus

· this is affinoid perfectoid

2). $GL_2(\mathbb{Q}_p) \curvearrowright |\mathcal{X}^*_{\Gamma(p^\infty)}|$

3). need to show that translates of $|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{anti}|$ by $GL_2(\mathbb{Q}_p)$

2) cover all of $|\mathcal{X}^*_{\Gamma}(p^\infty)|$



supersingular loci

· $GL_2(\mathbb{Q}_p)$-action preserves ordinary / supersingular parts

· not clear that you cover the entire supersingular locus

§2. The Hodge-Tate period map

· $|\pi_{HT}| : |\mathcal{X}^*_{\Gamma(p^\infty)}| \xrightarrow{GL_2(\mathbb{Q}_p)\text{-equivariant}} |\mathbb{P}^1|$

map on topological spaces.

· $\pi_{HT} : \mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}} \longrightarrow \mathbb{P}^1$

map of adic spaces.

$\rightsquigarrow$ Hodge-Tate filtration of $E$ is defined over $\mathbb{Q}_p$.

2). $\varepsilon \in (0, 1/2)$

$\pi_{HT}\left(\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\text{anti}}\right)$

contains an open nbhd $\mathcal{U}$ of $\mathbb{P}^1(\mathbb{Q}_p) \cap \mathcal{H}_2$.

3). <u>exercise</u>: see that

$GL_2(\mathbb{Q}_p) \cdot \mathcal{U}$ covers all of $|\mathbb{P}^1|$.

4). construct $\pi_{HT}$

$$\pi_{HT} : \mathcal{X}^*_{\Gamma(p^\infty)} \longrightarrow \mathbb{P}^1$$

by translating & gluing.

3) $\mathbb{P}^1 = \mathcal{F}\ell_1 \cup \mathcal{F}\ell_2$

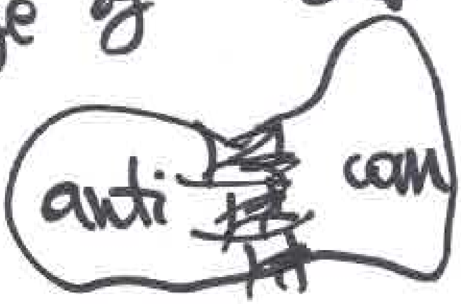
$s_1, s_2 \in H^0(\mathbb{P}^1, \mathcal{O}(1))$

$\mathcal{F}\ell_1 : |s_1| \geq |s_2|$

$\mathcal{F}\ell_2 : |s_2| \geq |s_1|$. $\qquad \chi^*$

$\mathcal{B}(p)$

<u>Claim</u>: 1). the image of

$\chi^*_{\Gamma(p^\infty)}(0)_{anti}$ is 


$\mathbb{P}^1(\mathbb{Q}_p) \cap \mathcal{F}\ell_2$

· key point: If $E/\mathcal{O}_C$ has ordinary reduction

$$\mathrm{Lie}\, E \otimes_C C(1) \subset T_p E \otimes_{\mathbb{Z}_p} C$$

$$\| \; C^2$$

is the line det by canonical subgp at each level in

$$\mathcal{G}/\mathcal{O}_C \longmapsto (T_p\mathcal{G}, \operatorname{Lie}\mathcal{G} \otimes_{\mathcal{O}_C} C$$

p div gp

Hodge-Tate filtration $\leftarrow \cap$

$T_p\mathcal{G} \otimes_{\mathbb{Z}_p} C(-1)$

$\wr\!\!\!\wr$

$C^2(-1)$

Add trivialization of $T_p\mathcal{G}$, e.g. ~~$\mathbb{Z}$~~

$$T_p\mathcal{G} \simeq \mathbb{Z}_p^2$$

$\rightsquigarrow$ flag variety $\mathbb{P}^1$.

<u>Natural construction</u>:

- there is a Newton stratification on $\mathbb{P}^1$ which matches$^*$ Newton stratification on $\mathcal{X}^*_{\Gamma(p^\infty)}$.

## §3. The geometry of $\pi_{HT}$:

$\mathbb{P}^1$ = good substitute for moduli of p-divisible gps of dim 1, height 2.

Scholze - Weinstein

(analogue of Riemann's classification of abelian varieties over $\mathbb{C}$)

$\exists$ equivalence of categories

$$\left\{ \begin{array}{c} p\text{-div. gps} \\ \text{over } \mathcal{O}_C \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} (T, W) \\ T \text{ free } \mathbb{Z}_p\text{-mod} \\ \text{of fin rank} \\ W \subset T \otimes_{\mathbb{Z}_p} C(-1) \end{array} \right.$$

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma(p^\infty)} & & \\ \downarrow & & \\ \mathcal{X}^*_{\Gamma} & \longrightarrow & \overline{X}^*_{\Gamma} \end{array}$$

specialization map

$\pi_{HT}$

$$\begin{array}{ccccc} \mathcal{X}^*_{\Gamma(p^\infty)} = & \mathcal{X}^{*,\mathrm{ord}}_{\Gamma(p^\infty)} & \sqcup & \mathcal{X}^{*,\mathrm{ss}}_{\Gamma(p^\infty)} \\ \downarrow & \downarrow \pi^{\mathrm{ord}}_{HT} & & \downarrow \pi^{\mathrm{ss}}_{HT} \\ \mathbb{P}^1 = & \mathbb{P}^1(\mathbb{Q}_p) & \sqcup & \Omega \end{array}$$

Drinfeld upper half plane

* closure relations on adic space reversed compared to special fiber

$*$ : Newton stratifications match on rank 1 points.

$$\mathcal{X}^{*,\mathrm{ord}}_{\Gamma(p^\infty)} = \overline{|\mathcal{X}^{*}_{\Gamma(p^\infty)}(0)}$$

closure.

What do fibers look like?

1).

$$\pi^{\mathrm{ord}}_{HT} : \mathcal{X}^{*,\mathrm{ord}}_{\Gamma(p^\infty)} \longrightarrow \mathbb{P}^1(\mathbb{Q}_p)$$

fibers are "perfectoid Igusa curves" $(\mathcal{I}g_{\Gamma,\infty})^{\mathrm{perf}}$

$\overline{X}_{\Gamma}(0)$ : has finite étale covers called Igusa curves

ordinary locus in $\overline{X}_{\Gamma}$

Igusa curves $Ig_{\Gamma, m}$, $m \in \mathbb{Z}_{\geq 1}$

obtained by trivializing

$$E_0[p^m]^{et} \simeq \left(\mathbb{Z}/p^m\mathbb{Z}\right)$$

$\rightsquigarrow$ $Ig_{\Gamma,\infty}$ (formal) scheme

over $\mathrm{Spf}\, \overline{\mathbb{F}_p}$

$\mathrm{Spec}\, \overline{\mathbb{F}_p}$

· take perfection of $Ig_{\Gamma,\infty}$

· ~~canon~~ canonical lift

to formal scheme

over $\mathrm{Spf}\, W(\overline{\mathbb{F}_p})$:

apply Witt vectors.

- take adic generic fiber of this formal scheme, base change to perfectoid field
- the tower $(\mathfrak{Ig}_{\Gamma,m})_m$ is well-understood (classical)

2). over supersingular locus:

$$\Omega = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p).$$

$$\pi_{HT}^{ss} : \mathcal{X}^{ss}_{\Gamma(p^\infty)} = \bigsqcup \mathcal{M}_{LT,\infty}$$

finitely many copies of

Lubin-Tate

deformation space of p-div grps

$$\pi_{HT} \downarrow$$

$$\Omega$$

$$\begin{tikzcd}
\bigsqcup \mathcal{M}_{LT,\infty} \arrow[r, "\sim"] \arrow[d, "\pi_{HT}"', "\substack{\text{Lubin-Tate} \\ p\text{-adic} \\ \text{étale}}"] & \bigsqcup \mathcal{M}_{Dr,\infty} \arrow[d, "\pi_{GM}", "\text{de Rham}"'] \\
\Omega \arrow[r, equal] & \Omega
\end{tikzcd}$$

fin. many copies

fin many copies

pro-finite étale

fibers: profinite sets.

## §4. Cohomology with torsion coefficients.

$$\begin{tikzcd}
& X^*_{\Gamma(p^\infty)} \arrow[dl] \arrow[dr, "\pi_{HT}"] & \\
X^*_{\Gamma} & & \mathbb{P}^1
\end{tikzcd}$$

$GL_2(\mathbb{Z}_p)$-cover on open modular curve

reduce to understanding $H^*_c(X_{\Gamma(p^\infty)}$

torsion

1). p-adic coeffs:

$$H^*_c(X_{\Gamma(p^\infty)}, \mathbb{Z}/p^n\mathbb{Z})$$

- primitive comparison theorem for $X^*_{\Gamma(p^\infty)}$ to move into coherent cohomology

$$\left\{\begin{array}{l} \cdot\ \pi^*_{HT}(\omega_1) \\ \cdot\ \pi^*_{HT}(\omega_2) \end{array}\right. \qquad \begin{array}{c} \omega_1, \omega_2 \\ \uparrow \\ H^0(\mathbb{P}^1, \mathcal{O}(1)) \end{array}$$

used to produced congruences

"fake Hasse invariants".

2). $\ell$-adic coefficients.

$$H^*_c(\mathcal{X}^*_{\Gamma(p^\infty)}, \mathbb{Z}/\ell^n\mathbb{Z})$$

perfectoid
modular
curve, infinite level
at $p$, $\ell \neq p$.

· using geometry
of $\pi_{HT}$

~~Consequences~~ Applications

Thm (in progress)

Thm : due to:

~~II~~. Allen, Calegari, C, Gee, Helm, Le Hung, Newton Scholze, Taylor, Thorne

I. If $E/F$, non-CM.
elliptic imaginary quadratic

then : 1). E is potentially automorphic

2). E satisfies Sato-Tate conjecture.

II. ~~Ra~~ Application to Ramanujan.