A = Huber ring

(A, A^+) = Huber pair

A will always be <u>complete</u>.

e.g. $A = \mathbb{Q}_p \langle T_1, \ldots T_n \rangle$

$X = \mathrm{Spa}(A, A^+)$

goals: - promote set to topological space

- equip it with a (pre)sheaf of rings
- form a locally ringed space
- consider sheaves of modules (e.g. vector bundles

  ↖ finite projective modules,

  coherent sheaves

  ↑

  finitely generated modules)

Also assume A is analytic ②

- topologically nilpotent elements of A ~~form~~ generate unit ideal (e.g. if A is Tate)

$\Updownarrow$

every valuation in $\mathrm{Spa}(A, A^+)$ is nontrivial.

$\Updownarrow$

the Banach open mapping theorem:

if $f: M \to N$ continuous ~~sur~~ surjection of complete first-countable topological A-modules

then f is a topological quotient map.

the topology defined by Huber on $Spa(A, A^+)$ is generated by certain "distinguished open" subsets

$$X\left(\frac{f_1 \cdots f_n}{g}\right) = \{v \in X : v(f_i) \leq v(g) \neq 0 \text{ for } i = 1, \dots, n\}$$

where $f_1 \cdots f_n, g$ generate <u>open</u> ideal in A

For A analytic
~~does~~ $f_1 \cdots f_n, g$ generate <u>unit</u> ideal

$\{v \in X : v(f_i) \leq v(g) \text{ for } i = 1, \dots, n\}$

This space is $Spa(B, B^+)$

$$B = A\langle T_1 \cdots T_n\rangle \Big/ \overline{(gT_1 - f_1, \dots gT_n - f_n)}$$

The map $(A, A^+) \to (B, B^+)$ ~~~
is <u>initial</u> among maps for which
$Spa(B, B^+)$ maps into $X\left(\frac{f_1 \cdots f_n}{g}\right)$.
$\to Spa(A, A^+)$

<u>rational localizations</u>

Also need to check:

$$\operatorname{Spa}(B, B^+) \xrightarrow{\sim} X\left(\frac{f_1 \cdots f_n}{g}\right) \subseteq \operatorname{Spa}(A, A^+)$$

homeomorphism.

Structure presheaf:

for $\mathcal{U} = X\left(\frac{f_1 \cdots f_n}{g}\right)$

take $\mathcal{O}(\mathcal{U})$ = the ring B from above

$$A\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle$$

$\Big\langle$ for general open $\mathcal{U}$

$$\mathcal{O}(\mathcal{U}) = \varprojlim B$$

over all $(A, A^+) \to (B, B^+)$

with $\operatorname{Spa}(B, B^+) \subseteq \mathcal{U}$ $\Big\rangle$

Tate: if $A = K\langle T_1, \ldots, T_n \rangle$

(Huber) for K = nonarchimedean field, or any quotient thereof,

then $\mathcal{O}$ is a sheaf.

Aaaah! The structure presheaf is not always a sheaf!

(Huber, Buzzard–Verberkmoes, Mihara)

A general reason for this:
if $\mathcal{O}$ is a sheaf, then
$(gT_1 - f_1, \dots, gT_n - f_n) \subseteq A\langle T_1, \dots, T_n\rangle$ is closed
whenever $f_1, \dots, f_n, g$ generate unit ideal.

If $\mathcal{O}$ is a sheaf, then may give $\mathrm{Spa}(A, A^+)$ the structure of a locally (v-)ringed space.

each stalk also carries a valuation.

Glue these to make adic spaces.

Huber*: if A is <u>strongly noetherian</u>
(i.e. $A\langle T_1 \cdots T_n\rangle$ is noetherian
for all $n \geq 0$)
then A is <u>sheafy</u> $\iff$ $\mathcal{O}$ is a sheaf.

includes classical affinoids
but not perfectoid rings
e.g. ~~$\mathbb{C}_p$~~ $\mathbb{C}_p\langle T^{1/p^\infty}\rangle$
$\cup$
$(T, T^{1/p}, T^{1/p^2}, \dots)$

Buzzard–Verberkmoes*:
if A is <u>stably uniform</u>
(i.e. for all rational localizations
$(A, A^+) \to (B, B^+)$, B is uniform)
$\Downarrow$
then A is sheafy. reduced

includes <u>reduced</u> affinoid algebras
and perfectoid rings

e.g. if A is perfectoid

then $A\langle T_1, \ldots, T_n \rangle$ is sheafy.

$$\Updownarrow$$

$$A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$$

splits in the category of topological

$A\langle T_1 \ldots T_n \rangle$

<u>Thm</u> (K-Liu) Rvochvai Liu

If A is sheafy, then

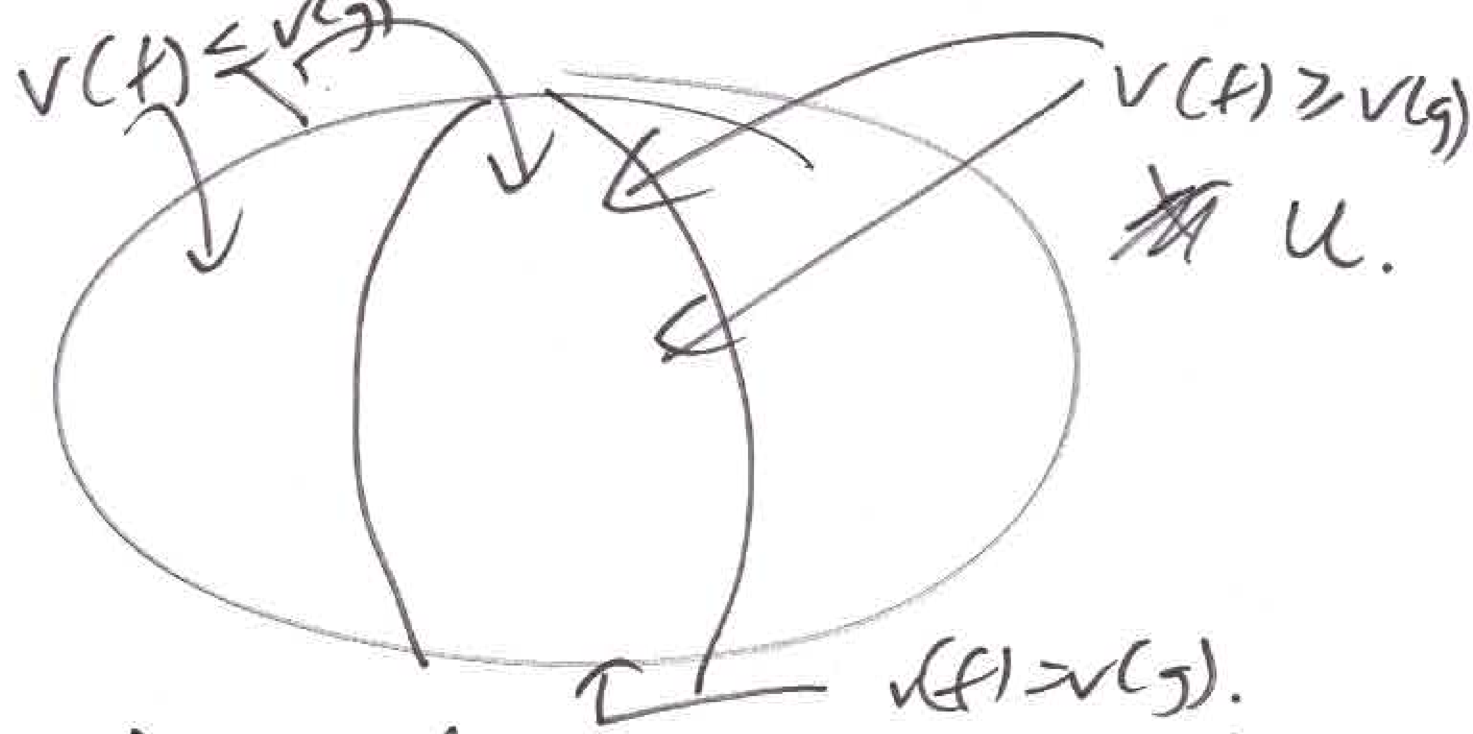
$H^i(U, \mathcal{O}) = 0$ for $U \subseteq X$ a rational subspace and $i > 0$.

$$\Uparrow$$

using Čech cohomology

observed by Tate ...

only need to verify vanishing of Čech cohomology in



$f, g \in \cancel{A}B$ generate unit ideal

Assume A is $\boxed{\text{sheafy}}$.

For M a finite projective A-module, let $\tilde{M}$ be associated sheaf

$\mathcal{U} = \mathrm{Spa}(B, B^+)$

$\tilde{M}(\mathcal{U}) = M \otimes_A B$.

$H^i(\mathcal{U}, \tilde{M}) = 0 \quad \forall\, \mathcal{U}$ rational, $i > 0$

<u>Thm</u> (K–Liu)

$$\left\{ \begin{array}{c} \text{finite projective} \\ A\text{-modules} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \underline{\text{vector bundle on } X} \\ \underline{\text{Sheaves of } \cancel{\mathcal{O}\text{-modules}}} \\ \mathcal{O}\text{-modules} \\ \underline{\text{locally}} \text{ of} \\ \text{form } \tilde{M}. \end{array} \right\}$$

$$H^0(X, \mathcal{F}) \longleftarrow \mathcal{F}$$

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If A strongly noetherian, can likewise equate finitely generated A-modules with coherent sheaves (Huber*)

Note: A noetherian
$\Longleftrightarrow$ every ideal I of A is closed.

K–Liu: can do something similar for finitely generated A-modules which are – complete for natural topology.
– pseudocoherent

(= admit projective resolution by finite proj modules)

$x \in X$

$$\mathcal{O}_x = \varinjlim \mathcal{O}(U)$$

U runs over rational subspaces containing X.