1)

Sheaves on Fargues-Fontaine curves.

(R, R^+) a perfectoid pair of char p

assume R is <u>Tate</u>

and fix a pseudouniformizer ϖ

$A_{inf} = W(R^+)$ "infinitesimal"

complete for the $(p, [\varpi])$-adic topology

A_{inf} is a Huber ring which is <u>not</u> analytic

$\{p = [\varpi] = 0\}$

$Spec(R^+/R^{\circ\circ}) \subset Spa(A_{inf}, A_{inf})$

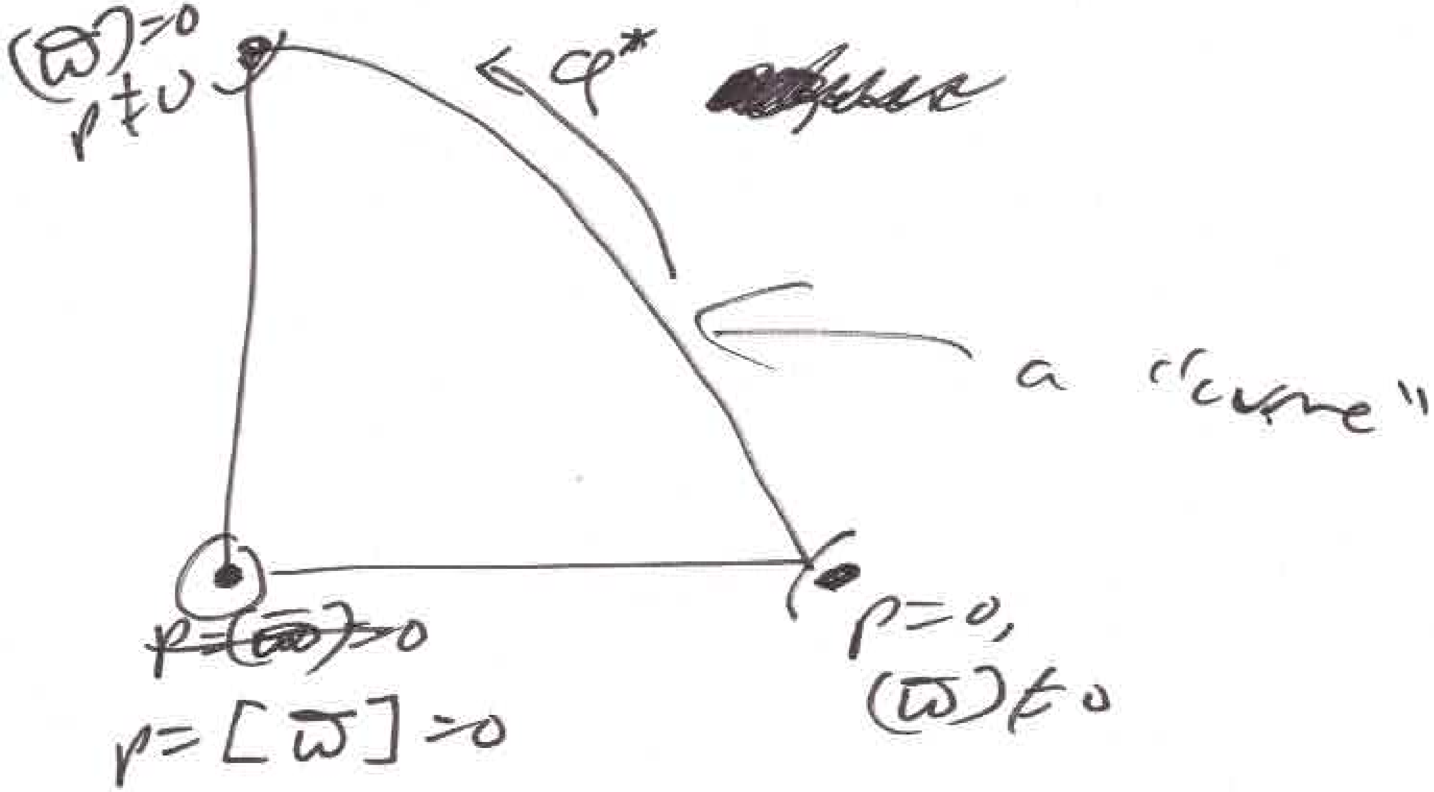
topologically nilpotent elements

e.g. if R is a perfectoid field

this is a single point.

in general, the complement of this set <u>is</u> analytic adic space.

2) Cartoon of $A_{\inf}$ (from Bhargav's notes)
if R is a field.



the analytic locus in this case is covered by adic spectra of strongly noetherian PID's.
(if remove $(\varpi)=0$).

now suppose I consider

$$Y_S = \{p, (\varpi)\neq 0\} \quad S=\mathrm{Spa}(R,R^+)$$

$\circlearrowleft \varphi$ free action!

3/

Quotient

$X_S := Y_S / \varphi^{\mathbb{Z}}$

$\parallel$

FF_S = adic Fargues-Fontaine curve $/S$.

This is an adic space b/c of strongly noetherian property.

These rings look like

$$A_{inf}[p^{-1}][[\varpi]^{-1}]\left\langle \frac{p^a}{[\varpi^b]}, \frac{[\varpi^c]}{p^d} \right\rangle$$

(strong analogy between A_{inf} and $\mathbb{Z}_p[[T]]$)

4)) Aside: same neighborhoods of $\begin{matrix} p \to 0 \\ \varpi \neq 0 \end{matrix}$

look like

$$\mathrm{Spa}\left(A_{\mathrm{inf}}[[\varpi]]^{-1} \left\langle \frac{p^a}{[\varpi^b]} \right\rangle, \dots \right)$$

adjoining $p^{p^{-\infty}}$ gives a natural example of a perfectoid ring which is Tate but not a $\mathbb{Q}_p$-algebra

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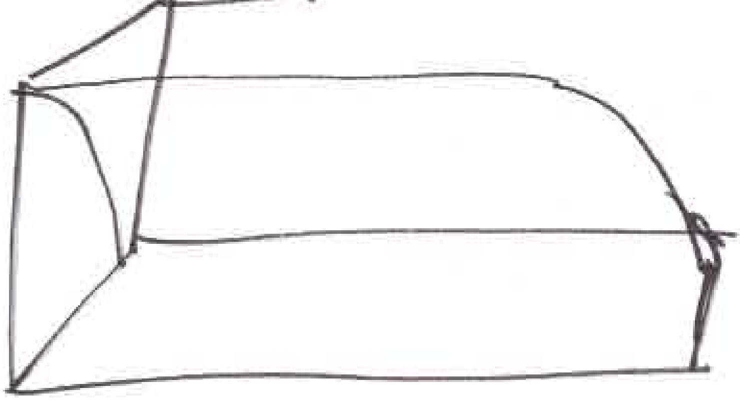
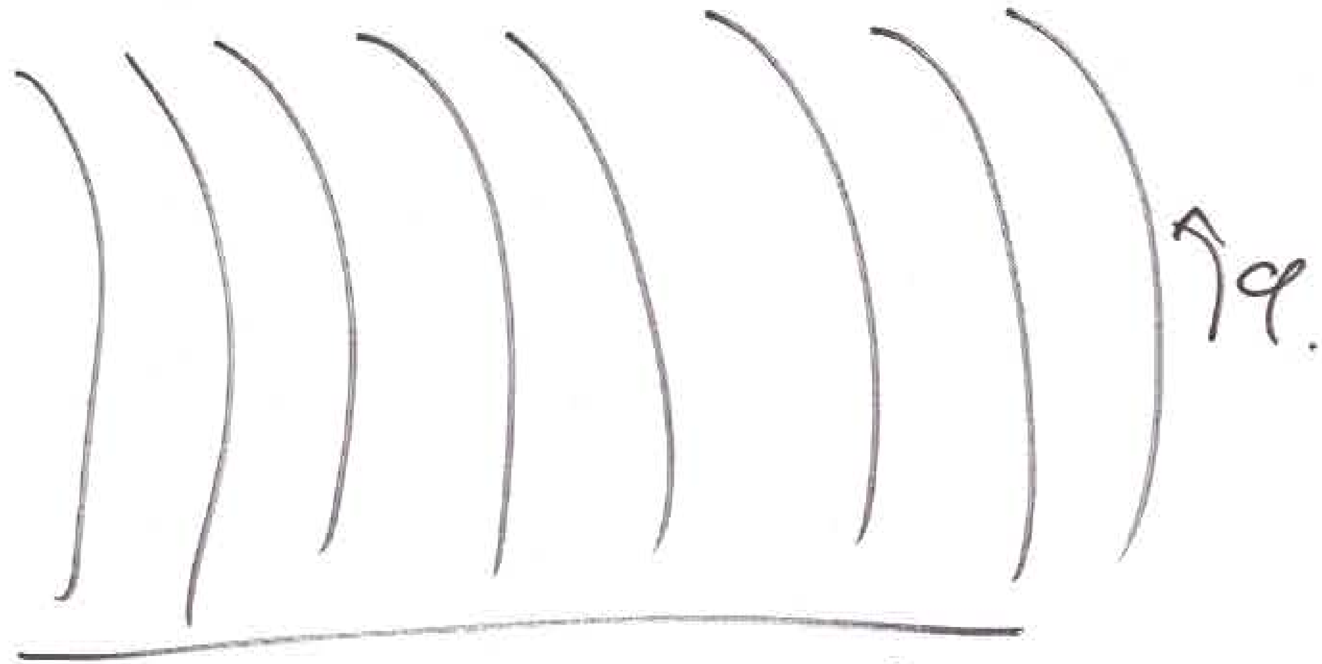
Relative case:
again, can consider analytic locus
(complement of $p = [\varpi] = 0$)

$Y_S = \{ p, [\varpi] \neq 0 \}$
$X_S = Y_S / \varphi$



5)

Y/S.



$S = \mathrm{Spa}(R, R^+)$

NOT A MAP OF ADIC SPACES!

$\downarrow$

adic relative FF curve/S

$FF_S = X_S$

picture is valid at level of Hausdorff quotient up to homotopy.

$\downarrow$

$S = \mathrm{Spa}(R, R^+)$.

6) given an untilt $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ of S, over $\mathbb{Q}_p$.

then get

$$\begin{tikzcd}
 & FF_S \arrow[d, dashed, "\text{topological}"] \\
S^\# \arrow[ur, hook, "\text{closed immersion}"] \arrow[r, dashed, leftrightarrow, "\text{topological}"'] & S \arrow[u, dashed, bend left]
\end{tikzcd}$$

"~~[illegible]~~ untilts of S over $\mathbb{Q}_p / \varphi$
$\sim$ sections of $FF_S \to S$"

7) for the rest of this hour, look at sheaves (esp. vector bundles) on FF_S.

Motivations:

- when $S =$ point, vector bundles relate to (φ, Γ)-modules which are used to analyze

$$\rho : G_F \longrightarrow GL_n(\mathbb{Q}_p).$$

picture will look like classical theory of vector bundles on algebraic curves / Rieman surfaces. (Geometric invariant theory = GIT).

- relativize to study $\pi_1(\ldots) \longrightarrow GL_n(\mathbb{Q}_p)!$

8) vector bundles on $FF_S := Y_S/\varphi^{\mathbb{Z}}$

stably uniform.
(by comparison to perfectoid)

$\updownarrow$

φ-equivariant vector bundles on Y_S.

e.g. $\mathcal{O}(1)$:= free bundle on one generator v

with $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(1)$

$1 \otimes v \mapsto p^{-1} v$.

normalization is set up so that

$H^0(FF_S, \mathcal{O}(-1)) = 0$

$H^0(FF_S, \mathcal{O}(1)) =$ huge.

for $R = C$ alg closed field, ~~Bach~~ Banach-Colmez space.

$B_C^{\varphi = p}$. $\quad B_C = H^0(Y_S, \mathcal{O})$

<u>Thm</u> (KL)
$\mathcal{O}(1)$ is <u>ample</u>: for every pseudocoherent sheaf $\mathcal{F}$

on ~~X~~ FF_S, $\mathcal{F}(n)$ is generated by global sections

for $n \gg 0$.

9)

One a point (i.e. R = field $R^{+} = R^{0}$)

$n \to \mathcal{O}(n)$

$\mathbb{Z} \to \mathrm{Pic}^{*}(FF_S) \xrightarrow{\deg} \mathbb{Z}$

$= \mathrm{Div}^{*}/\text{principal}$

$\downarrow$

degree of any vector bundle V

$\deg(V) := \deg\left(\bigwedge^{\mathrm{rank}(V)} V\right)$

$= \det(V)$

$\mu(V) := \dfrac{\deg(V)}{\mathrm{rank}(V)}$ slope of $V \neq 0$

V is <u>semistable</u> if V has no nonzero proper subbundle W with $\mu(W) > \mu(V)$.

10) K. $S = \mathrm{Spa}(R, R^+)$

<u>Thm</u> (Fargues–Fontaine) $F = R =$ field.

~~{ }~~ semistable VB on ~~X~~ FF_S of degree 0

$\updownarrow$

continuous representations of ~~Gal~~ G_F on fin.dim $\mathbb{Q}_p$-vector space.

$$V \longrightarrow \underset{G_F}{\circlearrowright} H^0(X_{\hat{F}}, V)$$

analogous to a theorem of Narasimhan–~~Seshadri~~ Seshadri on vector bundles on Riemann surfaces

$\updownarrow$

unitary π_1-representations.