Where do we go from here?

For example, Yves André has recently used perfectoid spaces to prove:

<u>Thm</u> (Direct Summand Conjecture, Hochster '73).

Let R regular ring, $R \hookrightarrow S$ finite. Then $R \hookrightarrow S$ has a splitting as R-modules.

($\Rightarrow$ descent along $R \to S$)

part of Hochster's "Homological Conjectures".

refined by Bhatt, Ma, Schwede, ...

develop theory of test ideals in mixed characteristic ... .

also: connections to algebraic topology
via topological Hochschild homology.

But for rest of talk, let's concentrate
on "mixed-characteristic shtukas".

<u>History of shtukas</u>: function fields:

Let $C/\mathbb{F}_q$ proj. smooth geom. connected curve.
Let $G/\mathbb{F}_q$ reductive group. (eg. $G = GL_2$.)
moduli space of shtukas / C with one leg

$$f : \mathrm{Sht}_{\cdots}^{\cdots} \longrightarrow C.$$

(analogue of Shimura varieties
$$\mathrm{Sh} \longrightarrow \mathrm{Spec}\, \mathbb{Z}.)$$

$$R^i f_* \overline{\mathbb{Q}}_\ell \circlearrowleft \pi_1(C) = \mathrm{Gal}(\bar{F}/F)^{\mathrm{unr}}.$$

$$G(\mathbb{A}). \quad \mathbb{A} = \mathbb{A}_F = \text{adèles of } F.$$

F = function field of C

Thm. (Drinfeld, L. Lafforgue ...).

$$R^i f_* \overline{\mathbb{Q}}_\ell = \bigoplus_{\substack{\text{certain}\\ \text{automorphic}\\ \text{repr. } \pi\\ \text{of } G(\mathbb{A})}} \pi \otimes \sigma(\pi).$$

$\sigma(\pi) \circlearrowleft \mathrm{Gal}(\overline{F}/F)$.

This association

$$\left\{ \begin{array}{c}\text{autom repr.}\\ \text{of } G(\mathbb{A})\end{array} \right\} \longrightarrow \left\{ \text{Galois repr.} \right\}$$

$$\pi \longmapsto \sigma(\pi).$$

defines the global Langlands correspondence (in some cases).

Unfortunately, not all automorphic π.

Insight of Drinfeld: Can get all π if one looks at spaces of shtukas with two legs.

2 legs:

$$f : \mathrm{Sht}^{\dots}_{\dots} \longrightarrow C \times C.$$

$$\underset{\underset{G(\mathbb{A})}{\circlearrowleft}}{R^i f_* \overline{\mathbb{Q}_\ell}} \circlearrowleft \pi_1\left(C \times C/_{\varphi^{\mathbb{Z}}}\right) \cong \pi_1(C) \times \pi_1(C).$$

Drinfeld's lemma.

Thm (same people). (for good choices of data).

$$R^i f_* \overline{\mathbb{Q}_\ell} = \bigoplus_{\substack{\text{all } \pi \\ \text{"cuspidal" autom. repr. of } G(\mathbb{A})}} \pi \otimes \underset{\underset{\pi_1(C)}{\circlearrowleft}}{\sigma(\pi)} \otimes \underset{\underset{\pi_1(C).}{\circlearrowleft}}{\sigma(\pi)^\vee}$$

Get global Langlands correspondence for

GL_2 : Drinfeld

GL_n : L. Lafforgue

any G : V. Lafforgue

We would love to do the same over number fields.

Obvious Problem: What is the analogue of $C \times_{\mathbb{F}_q} C$?

Magic of diamonds: Can make sense not of $\operatorname{Spec} \mathbb{Z} \times_{\mathbb{F}_1} \operatorname{Spec} \mathbb{Z}$, but at least of $\operatorname{Spec} \mathbb{Q}_p \times_{\mathbb{F}_1} \operatorname{Spec} \mathbb{Q}_p$.

(or even $\underbrace{\operatorname{Spec} \mathbb{Z}_p \times \operatorname{Spec} \mathbb{Z}_p}_{\text{completion at } (p,p)}$.)

Namely, can take product

$$\operatorname{Spd} \mathbb{Q}_p \times \operatorname{Spd} \mathbb{Q}_p$$

in category of diamonds, get something "2-dim'l".

$$\mathrm{Spd}\,\mathbb{Q}_p \times \mathrm{Spd}\,\mathbb{Q}_p$$
$$\|$$
$$\mathrm{Spd}\,\mathbb{Q}_p \times \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}} / \underline{\mathbb{Z}_p^*}$$
$$\|$$
$$\mathrm{Spd}\,\mathbb{Q}_p \times \mathrm{Spd}\,\mathbb{F}_p((t^{1/p^\infty})) / \underline{\mathbb{Z}_p^*}$$
$$\|$$
$$\cancel{\mathrm{Spd}}\ \left(\widetilde{\mathbb{D}}^*_{\mathbb{Q}_p}\right)^\diamond / \underline{\mathbb{Z}_p^*}.$$

perfectoid punctured open unit disc $/\mathbb{Q}_p$.

analogue of Drinfeld's lemma:

__Thm.__ $\pi_1\left(\mathrm{Spd}\,\mathbb{Q}_p \times \mathrm{Spd}\,\mathbb{Q}_p / \varphi^{\mathbb{Z}}\right)$
$$\cong$$
$$\pi_1(\mathrm{Spd}\,\mathbb{Q}_p) \times \pi_1(\mathrm{Spd}\,\mathbb{Q}_p)$$
$$\|$$
$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p).$$

equivalently:

$$\pi_1\left(\widetilde{\mathbb{D}}^*_{\mathbb{Q}_p} / \underline{\mathbb{Q}_p^*}\right) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^2.$$

$$\underline{\mathbb{Q}_p^*} \cong \underline{\mathbb{Z}_p^*} \times \varphi^{\mathbb{Z}}, \quad \varphi^{\mathbb{Z}} \leftrightarrow p^{\mathbb{Z}}.$$

or: $\pi_1\left(\widetilde{\mathbb{D}}^*_{\mathbb{C}_p} / \underline{\mathbb{Q}_p^*}\right) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

moduli spaces of shtukas: local, mixed-char.

with one leg:

$$\mathrm{Sht}_{\dots}^{\dots} \longrightarrow \mathrm{Spd}\, \mathbb{Q}_p.$$

These turn out to be (generalizations of) Rapoport-Zink spaces (local p-adic analogues of Shimura varieties).

Example (Lubin-Tate spaces).

Let $H/\overline{\mathbb{F}}_p$ 1-dim'l formal group of height n.

($\Rightarrow$ is p-div. group)

deformation space of H:

$$\mathfrak{X}_H \cong \operatorname{Spf} W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]].$$

generic fibre $\mathcal{M}_H$ $(n-1)$-dim'l open unit disc.

tower: $\dots \longrightarrow \mathcal{M}_{H,1} \to \mathcal{M}_{H,0} = \mathcal{M}_H$

$\mathcal{M}_{H,m}$ classifies isom.

$$\mathcal{H}[p^m] \cong (\mathbb{Z}/p^m\mathbb{Z})^n,$$

$\mathcal{H}$ = univ. deformation of H.

$$\mathcal{M}_{H,\infty} \sim \varprojlim_m \mathcal{M}_{H,m}.$$

perfectoid space. (S.-Weinstein).

Thm (S.-Weinstein). Let $C/\mathbb{Q}_p$ alg closed complete extension,

let $\underset{\infty \in}{FF}_{C^\flat}$ Fargues - Fontaine corr. to $C^\flat$.

Then:

$$\mathcal{M}_{H,\infty}(C) = \left\{ \begin{array}{l} \mathcal{O}^n \xhookrightarrow{f} \mathcal{O}(1/n) \\ \text{s.th. coker } f \text{ is supported at } \infty \end{array} \right\}.$$

This can also be said in terms of shtukas "with one leg at ∞".

Several legs: There is no obstruction to considering moduli spaces of shtukas with any number of legs:

Test objects: $S \in \mathrm{Pfd} = \{$perfectoid spaces of char. $p\}$.

legs at S-valued points $x_1, \dots, x_n : S \to \mathrm{Spd}\, \mathbb{Q}_p$ of $\mathrm{Spd}\, \mathbb{Q}_p$.

These correspond to untilts

$$S_1^\#, \dots, S_n^\# \quad \text{of } S.$$

$$\begin{tikzcd} \text{graph of } x_i : S \to \operatorname{Spd}\mathbb{Q}_p \arrow[r, hook, "\substack{\text{closed}\\\text{immersion}\\\text{of adic spaces}}"] & Y_S \text{ "} = S \times \operatorname{Spd}\mathbb{Q}_p\text{"} \\ S_i^\# \arrow[u, equal] \arrow[r, dashed, "\sim"] & S \end{tikzcd}$$

Can consider φ-modules over Y_S (or compactification of it) with poles/zeroes at the divisors

$$S_i^\# \hookrightarrow Y_S.$$

$$\rightsquigarrow \quad f : \mathrm{Sht}^{\dots}_{\dots} \longrightarrow \operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p.$$

$$\begin{array}{c} R^i f_* \overline{\mathbb{Q}_\ell} \circlearrowleft \pi_1\left(\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p/\varphi^{\mathbb{Z}}\right) \\ \circlearrowleft \qquad\qquad\qquad \parallel \\ G(\mathbb{Q}_p) \qquad\qquad \text{"}\,\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^2\text{"}. \end{array}$$

Thm (Hesselholt).

$$\pi_0\left(THH(\mathcal{O}_{\mathbb{C}})^{\wedge}_{p}\right)^{hS^1} = A_{\inf}.$$