Historic Remarks about genesis of paper "Perfectoid Spaces".

(Or: Why perfectoid spaces are a failed theory.)

In 2007, I came to Bonn as undergrad, studied under M. Rapoport.

He gave me the following problem to think about:

Weight-Monodromy Conjecture

Let X smooth projective scheme $/\mathbb{Q}_p$.

Fix $i \geq 0$, $\ell \neq p$ prime.

Consider the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation

$$V = H^i_{et}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$$

<u>known:</u> • There is a weight decomposition:

If $\Phi \in Gal(\bar{\mathbb{Q}}_p / \mathbb{Q}_p)$ geometric Frobenius,

then

$$V = \bigoplus_{j=0}^{2i} V_j$$

where Φ acts through Weil numbers of weight j on V_j.

(Rapoport - Zink ~1980 if X has semistable reduction

de Jong ~1995 in general (reduction to semistable case))

• There is a monodromy operator

$$N : \quad V \longrightarrow V(-1).$$

Tate twist.

coming from action of inertia subgroup.

In particular, $N : V_j \longrightarrow V_{j-2}$.

Then: $\boxed{\forall_{j=0,\dots,i} : \ N^j : V_{i+j} \xrightarrow{\sim} V_{i-j}.}$

<u>Examples</u>. 1). If X has good reduction, i.e.

$\exists$ smooth projective $\mathfrak{X}/\mathbb{Z}_p$ with generic fibre X,

then
$$V \cong H^i_{et}(\mathfrak{X}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell).$$
$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p). \end{array}$$

so inertia acts trivially.

$\Rightarrow$ $N = 0$.

$\forall j > 0$ $N^j = 0$: $V_{i+j} \xrightarrow{\sim} V_{i-j}$

so equiv., $V_j = 0$ $\forall j \neq i$.

i.e. $V = V_i$.

But this follows from Weil conjectures for $\mathfrak{X}_{\mathbb{F}_p}$.

2). If $X = E$ elliptic curve with multiplicative reduction. "$E = \mathbb{G}_m / q^{\mathbb{Z}}$" $0 \neq q \in \mathbb{Q}_p$, $|q| < 1$.

as rigid-analytic/adic spaces.

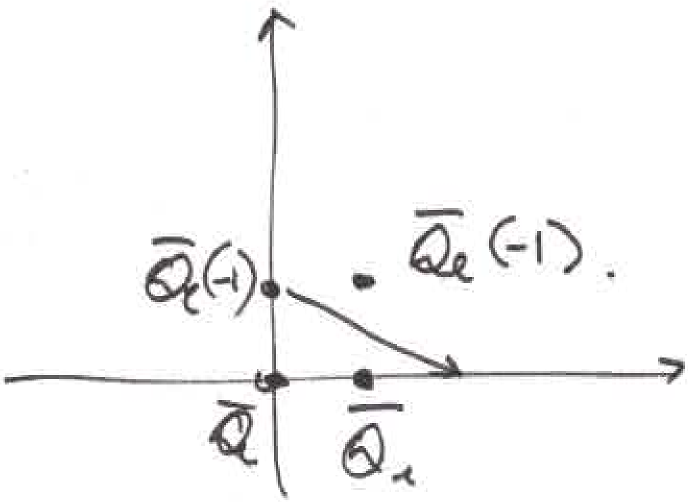
Then

$$H^1_{ét}(E_{\bar{\mathbb{Q}}_p}, \bar{\mathbb{Q}}_\ell) = H^1_{ét}(\mathbb{G}_{m,\bar{\mathbb{Q}}_p}/q^{\mathbb{Z}}, \bar{\mathbb{Q}}_\ell).$$

Then by Hochschild-Serre, have spectral seq.

$$H^i\Big(\mathbb{Z}, \underbrace{H^j_{ét}(\mathbb{G}_{m,\bar{\mathbb{Q}}_p}, \bar{\mathbb{Q}}_\ell)}_{= \begin{cases} \bar{\mathbb{Q}}_\ell & j=0 \\ \bar{\mathbb{Q}}_\ell(-1) & j=1 \\ 0 & \text{else.} \end{cases}}\Big) \Rightarrow H^{i+j}_{ét}(\mathbb{G}_{m,\bar{\mathbb{Q}}_p}/q^{\mathbb{Z}}, \bar{\mathbb{Q}}_\ell).$$

with trivial $\mathbb{Z}$- action.



So $0 \to \bar{\mathbb{Q}}_\ell \to H^1_{ét}(E_{\bar{\mathbb{Q}}_p}, \bar{\mathbb{Q}}_\ell) \to \bar{\mathbb{Q}}_\ell(-1) \to 0$

So $V_2 = \bar{\mathbb{Q}}_\ell(-1)$, $V_0 = \bar{\mathbb{Q}}_\ell$

(splitting $V = V_0 \oplus V_2$ depends on choice of Φ;

Weight - Monodromy predicts

$$N: V_2 \xrightarrow{\sim} V_0.$$

can be checked by hand: Use that inertia action is nontrivial on ℓ-power roots of q. for $i=1,2$.

Remarks. 1). Conjecture is known ~~in dim 1 and 2~~.

(dim 1: ~~uses~~ reduce to abelian varieties or curves, use Néron models / semistable models).

(dim 2: Rapoport - Zink.)
+ de Jong.

2). Known in equal characteristic p, i.e. over $\mathbb{F}_p((t))$.

proved in Deligne's Weil 2 paper, uses that L-function over function fields have good properties.

3). Conversely, weight-monodromy conj. is critical to understanding local factors of Hasse-Weil zeta functions at places of bad reduction.

($\Longleftarrow$ the Hasse-Weil zeta function "has no poles in region of absolute convergence".)

Rapoport's suggestion: Try to reduce to case of equal characteristic after base change to some very ramified $K/\mathbb{Q}_p$.

Idea: If $\mathfrak{X}/\mathcal{O}_K$ integral (semistable, say) model of $X \times_{\mathbb{Q}_p} K$, then $\mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/p$ lives over $\mathcal{O}_K/p \cong \mathbb{F}_q[t]/t^e$, ~~e = [K:Q_p]~~ e = ramification index of $K/\mathbb{Q}_p$.

"If $e \gg 0$, this is almost $\mathbb{F}_p[[t]]$."

Of course, this does not really work;
as even if e is large, still not to
deform

$$\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/p \quad \text{from} \quad \mathcal{O}_K/p = \bar{\mathbb{F}}_p[t]/t^e$$

to $\bar{\mathbb{F}}_p[[t]]$.

usually, there are (a lot of) obstructions.

Also, in the end need to relate

$$V = H^i_{ét}(X_{\bar{\mathbb{Q}}_p}, \bar{\mathbb{Q}}_\ell) \circlearrowleft \operatorname{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$$

to $H^i_{ét}(X'_{\overline{\bar{\mathbb{F}}_p((t))}}, \bar{\mathbb{Q}}_\ell) \circlearrowleft \operatorname{Gal}(\bar{\mathbb{F}}_p((t))^{sep}/\bar{\mathbb{F}}_p((t)))$,

where $X'/\bar{\mathbb{F}}_p((t))$ is generic fibre of deformation.

In semistable case, can use log-geometry to do this.

(related to isomorphism of tame quotients of
$\operatorname{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\operatorname{Gal}(\bar{\mathbb{F}}_p((t))^{sep}/\bar{\mathbb{F}}_p((t)))$.)

Turning these ideas in my head, I read

<u>Thm</u> (Fontaine - Wintenberger).

$$\mathrm{Gal}\left(\overline{\mathbb{Q}_p} / \mathbb{Q}_p(p^{1/p^\infty})\right) \cong \mathrm{Gal}\left(\mathbb{F}_p((t))^{sep} / \mathbb{F}_p((t))\right)$$

canonically.

Proof involves Fontaine's construction like

$$\varprojlim_{\text{Frob}} \mathcal{O}_{\overline{\mathbb{Q}_p}} / p .$$

hard to understand what it means.

Later, I learned from Faltings that:

<u>Thm</u>.

$$\pi_1^{\text{ét}}\left(\mathrm{Spec}\, \mathbb{Q}_p(p^{1/p^\infty})\langle T^{\pm 1/p^\infty}\rangle\right)$$

$$\cong$$

$$\pi_1^{\text{ét}}\left(\mathrm{Spec}\ \mathbb{F}_p((t))\langle T^{\pm 1}\rangle\right).$$

Things started to resolve after I realized the following proof of Fontaine-Wintenberger's Thm:

$\{\text{finite étale } \mathbb{Q}_p(p^{1/p^\infty})\text{-alg}\}$

$\cong$ Tate "p-div. groups"

$\{\underline{\text{almost}} \text{ finite étale } \mathbb{Z}_p[p^{1/p^\infty}]\text{-alg.}\}$ in technical sense

$\cong$ unique lifting of fin. étale alg.

$\{\text{almost finite étale } \mathbb{Z}_p[p^{1/p^\infty}]/p\text{-alg.}\}$

$=$

$\{\text{almost finite étale } \mathbb{F}_p[t^{1/p^\infty}]/t\text{-alg}\}$.

$\cong$

$\{\text{almost finite ét. } \mathbb{F}_p[[t]][t^{1/p^\infty}]\text{-alg}\}$

$\cong$ Tate.

$\{\text{finite étale } \mathbb{F}_p((t))(t^{1/p^\infty})\text{-alg}\}$

$\cong$

$\{\text{ -"- } \mathbb{F}_p((t))\text{-alg}\}$

This suggested what to do in relative case. Find some notion of "perfectoid"

$$\{\text{perfectoid } \mathbb{Q}_p(p^{1/p^\infty})\text{-alg}\}.$$

$$\cong$$

$$\{\text{perfectoid almost } \mathbb{Z}_p[p^{1/p^\infty}]\text{-alg}\}.$$

$$\cong$$

$$\left\{\begin{matrix}\text{perfectoid} \\ \text{almost} \\ \mathbb{F}_p[t^{1/p^\infty}]/t\text{-alg.}\end{matrix}\right\} = \{\text{perfectoid almost } \mathbb{Z}_p[p^{1/p^\infty}]/p\text{-alg.}\}.$$

$$\cong$$

$$\vdots$$

$$\cong$$

$$\{\text{perfectoid } \mathbb{F}_p((t))(t^{1/p^\infty})\text{-alg.}\}.$$

need unique lifting property:

If R perfectoid (almost) $\mathbb{Z}_p[p^{1/p^\infty}]/p$-alg., then

cotangent complex. $\longrightarrow$ $\mathbb{L}_{R/(\mathbb{Z}_p[p^{1/p^\infty}]/p)} = 0.$

<u>Lemma</u> (Gabber - Ramero). If $S \to R$ map of $\mathbb{F}_p$-algebras that is "relatively perfect", i.e. relative Frob

$$\Phi_{R/S} : R \otimes_{S, \varphi} S \xrightarrow{\sim} R.$$

is isomorphism,

then $\mathbb{L}_{R/S} \simeq 0$.

(<u>Proof</u>. $\Phi_{R/S}$ isom. of $\mathbb{L}_{R/S}$, but also equal to 0,

as $d(x^p) = p\,x^{p-1}\,dx = 0$.)

<u>Definition</u>. A perfectoid $\mathbb{Q}_p(p^{1/p^\infty})$-alg. is a uniform Banach $\mathbb{Q}_p(p^{1/p^\infty})$-alg R s.th.

$$(R^\circ/p) / (\mathbb{Z}_p[p^{1/p^\infty}]/p).$$

is relatively perfect, where

$R^\circ = \{$powerbounded elements in $R\}$.

equivalently: $\Phi : R^\circ/p \longrightarrow R^\circ/p$ is surjective.

$$x \longmapsto x^p$$

<u>Corollary</u>.

$$\begin{array}{ll} R & \{\text{perfectoid } \mathbb{Q}_p(p^{1/p^\infty})\text{-alg.}\}. \\ \downarrow & \qquad\qquad \wr\wr \\ R^\flat & \{\text{perfectoid } \mathbb{F}_p((t))(t^{1/p^\infty})\text{-alg.}\} \end{array}$$

This can be made explicit in terms of Fontaine's functor:

$$R^\flat = \varprojlim_{\text{Frob}} \left(R^\circ/p\right) \underset{\mathbb{F}_p[[t]][t^{1/p^\infty}]}{\otimes} \mathbb{F}_p((t))(t^{1/p^\infty}).$$

$\vdots$

Pass to geometry:

<u>Corollary</u>. $$\left(\mathbb{P}^{n,ad}_{\mathbb{F}_p((t))}\right)_{ét} = \varprojlim_{\varphi} \left(\mathbb{P}^{n,ad}_{\mathbb{Q}_p(p^{1/p^\infty})}\right)_{ét}.$$

$$\varphi(x_0 : \ldots : x_n) = (x_0^p : \ldots : x_n^p).$$

Now, $X \subset \mathbb{P}^n_{\mathbb{Q}_p}$ is your smooth proj. variety.

$$
\begin{tikzcd}
\mathbb{P}^n_{\mathbb{F}_p((t))} \arrow[r, "\pi"] & \mathbb{P}^n_{\mathbb{Q}_p(p^{1/p^\infty})} \\
\pi^{-1}\left(X_{\mathbb{Q}_p(p^{1/p^\infty})}\right) \arrow[r] \arrow[u, phantom, sloped, "\subset"] & X_{\mathbb{Q}_p(p^{1/p^\infty})} \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}
$$

Apply Deligne to .

Problem: This is not algebraic!

But how far can it be away from algebraic?

Easy case. If X complete intersection, then in any ε-neighborhood of $\pi^{-1}(X_{\mathbb{Q}_p(p^{1/p^\infty})})$, there are algebraic varieties of same dim. enough to conclude