Perfectoid Fields

Def. let K is an <u>nonarch</u> <u>field</u> (complete) of res. char. p. $|\cdot| : K \to \mathbb{R}_{\geq 0}$.

K is a perfectoid field if

(a) $|K^\times|$ is nondiscrete

(b) $\Phi : K^\circ/p \to K^\circ/p$ is surjective.
$$x \mapsto x^p$$

Rmk. If $\operatorname{char} K = p$, (b) says K is perfect.

K perfectoid $\Leftrightarrow$ K perfect.

Eg $K = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$, $K^\circ = \mathbb{Z}_p[\mu_{p^\infty}]^\wedge$

$K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$

$K = \mathbb{Q}_p(E[p^\infty])^\wedge$. $E/\mathbb{Q}_p$ ec.

$K = \mathbb{F}_p((t^{1/p^\infty}))^\wedge$.

Lemma $|K^\times|$ is p-divisible.

Pf. Let $x \in K^\circ$ $|p| < |x| < 1$. (by (a)),
by (b) $\exists\, y \in K^\circ$, $|y^p - x| < |p| \Rightarrow |y| = |x|^{1/p}$.
$|p| = |x| \left|\frac{p}{x}\right|$. $\square$

Tilts

② Given a perfectoid field K

$$K^\flat = \varprojlim_{x \mapsto x^p} K$$

$$= \{(x_0, x_1, \ldots) \mid x_i \in K, \ x_i^p = x_{i-1}\}$$

addition law:

$$(x_i) + (y_i) = (z_i)$$

$$z_i = \lim_{m \to \infty} (x_{i+m} + y_{i+m})^{p^m}$$

makes $K^\flat$ into a field.

Write $K^\flat \to K$ hom. of mult monoids

$(x_0, x_1, \ldots) = f \mapsto f^\# = x_0$

For $f \in K^\flat$, let $|f|' = |f^\#|$

In fact $K^\flat$ is a perfectoid field of char p.

To see this,

$$\begin{tikzcd}
\varprojlim K^\circ \arrow[d, "\substack{\text{isom.}\\ \text{of mult}\\ \text{monoids}}"'] \arrow[r, phantom, "\subseteq"] & K^\flat = \varprojlim K \\
\underset{x \mapsto x^p}{\varprojlim} K^\circ/p &
\end{tikzcd}$$

perfect ring $\rightarrow$ $\varprojlim_{x \mapsto x^p} K^\circ/p$

$$K^\flat = (\varprojlim K^\circ)\left[\tfrac{1}{\varpi}\right]$$

$0 < |\varpi| < 1$

May chose ϖ so that $|\varpi| = |\varpi^\#| = |p|$

③

Eg. $\cdot$ $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$

$$K^\circ = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$$

$$= \mathbb{Z}_p[\![T^{1/p^\infty}]\!]/(T-p)$$

$$K^\circ/p = \mathbb{F}_p[T^{1/p^\infty}]/(T)$$

$$K^{\flat\circ} = \varprojlim_{\Phi} K^\circ/p = \varprojlim_{\Phi} \mathbb{F}_p[T^{1/p^\infty}]/(T)$$

$$\simeq \varprojlim_{id} \mathbb{F}_p[T^{1/p^\infty}]/(T^{p^n})$$

$$\simeq: \mathbb{F}_p[\![T^{1/p^\infty}]\!]$$

$$K^\flat = \mathbb{F}_p(\!(t^{1/p^\infty})\!). \qquad t = (p, p^{1/p}, p^{1/p^2}, \ldots)$$

$$t^\# = p$$

$\cdot$ $K = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$

$$1, \zeta_p, \zeta_{p^2}, \ldots \in K$$

$$t = (0, 1-\zeta_p, 1-\zeta_{p^2}, \ldots) \in \varprojlim_{\Phi} K^\circ/p \cong K^{\flat\circ}$$

$$t^\# = \lim_{n\to\infty} (1-\zeta_{p^n})^{p^n}$$

$$K^\flat \simeq \mathbb{F}_p(\!(t^{1/p^\infty})\!).$$

④.

$\mathbb{F}_p((t^{1/p^\infty}))$ is contained in any perfectoid field of char p.

<u>Thm</u> (Tilting Equivalence) Let K be perfectoid. For L/K finite separable, L is also perfectoid, and $L^\flat/K^\flat$ is finite separable, $[L^\flat : K^\flat] = [L : K]$. $L \mapsto L^\flat$ is an equivalence, and therefore

$$\mathrm{Gal}(K^{sep}/K) \simeq \mathrm{Gal}(K^{\flat\, sep}/K^\flat).$$

<u>Eg.</u>

$$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p} = \widehat{\overline{K}}, \quad K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$$

$$\mathbb{C}_p^\flat \simeq \widehat{\overline{K^\flat}} = \widehat{\overline{\mathbb{F}_p((t^{1/p^\infty}))}}$$

<u>Inverse</u>? Given $L/K^\flat$ how to find $L^\#/K$, $L^{\#\flat} = L$?

For a perfect ring R of char p, the Witt ring $W(R)$ is p-adically complete, $W(R)/p = R$, $\exists\ R \to W(R)$, $a \mapsto [a]$, st $[a] \bmod p = a$.

$$W(R) = \{[a_0] + [a_1]p + \dots \mid a_i \in R\}$$

$$W(\mathbb{F}_p) = \mathbb{Z}_p$$

If K is perfectoid char 0, (5)

$$\theta_K : W(K^{\flat\circ})[\tfrac{1}{p}] \longrightarrow K$$

$$\sum_{n \gg -\infty} [a_n] p^n \longmapsto \sum_{n \gg -\infty} a_n^{\#} p^n$$

is a surjective ring hom, whose kernel is

(ξ_K), $\quad \xi_K = \underbrace{[\varpi] + \alpha p}_{\substack{\text{primitive} \\ \text{degree } 1}}$, $\quad$ $\varpi \in K^\flat$ is a pseudo-unif $\alpha \in W(K^{\flat\circ})^*$.

Eg

$$K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$$

$$t \in K^\flat = \mathbb{F}_p((t^{1/p^\infty}))$$

$$t^{\#} = p$$

$$\theta_K([t]) = p$$

$$\xi_K = [t] - p.$$

Eg

$$K = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$$

$$K^\flat \ni \varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots), \quad \varepsilon^{\#} = 1$$

$$\theta_K([\varepsilon]) = 1$$

$$\xi_K = \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} = 1 + [\varepsilon^{1/p}] + \ldots + [\varepsilon^{\frac{p-1}{p}}]$$

⑥ If $L/K^\flat$ finite,

$$L^\# = W(L^\circ) \underset{W(K^{\flat\circ}),\theta_K}{\otimes} K$$

Given K perfectoid char p, what are all "untilts" to char 0?

<u>Thm.</u> $\{(\xi) \subseteq W(K^\circ),\ \xi \text{ primitive deg } 1\} \Rightarrow \{\text{untilts to char } 0\}$

$$(\xi) \mapsto \frac{W(K^\circ)[\frac{1}{p}]}{(\xi)}$$

Assume $K = C$ is alg. closed

let $C^\#$ be untilt to char 0

$$1, \zeta_p, \zeta_{p^2}, \ldots \in C^\#$$

$$(1, \zeta_p, \zeta_{p^2}, \ldots) \in C^{\#\flat} = C$$

$$\in 1 + m_C \leftarrow \text{~~group~~ } \mathbb{Z}_p\text{-module}$$

let $H = \hat{\mathbb{G}}_m$, formal mult group $/\mathbb{Z}_p$

$$H(C^\circ) = 1 + m_C \text{ as a } \cancel{\mathbb{Z}_p\text{-module}}\ \mathbb{Q}_p\text{-vector space}$$

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Thm. (Fargues - Fontaine)

$$\left\{\begin{matrix}\text{untilts of } C \\ \text{to char } 0\end{matrix}\right\} \xrightarrow{\sim} \left(H(C^\circ) \setminus \{0\}\right)/\mathbb{Z}_p^*.$$

$$C^\# \longrightarrow \varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in H(C^\circ)$$

$$W(C^\circ)[\tfrac{1}{p}]/(\xi) \longleftarrow\!\shortmid\ \varepsilon \in H(C^\circ) \setminus \{0\}$$

$$\xi = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = 1 + [\varepsilon^{1/p}] + \ldots [\varepsilon^{\frac{p-1}{p}}]$$

K char p

$$K^\flat = \varprojlim_{x \mapsto x^p} K = K$$

$$\{\text{untilts of } C\} \xrightarrow{\sim} H(C^\circ)/\mathbb{Z}_p^*$$

$$H(C^\circ) = 1 + m_C \supseteq \cancel{1 + m_C^2 = m}$$