X algebraic variety $/\mathbb{F}_q$.

Def 1:

$$\zeta_X(s) := \prod_{x \in X} \frac{1}{1 - |k(x)|^{-s}}$$

G-L $\Updownarrow$

Def 2

$$\zeta_X(s) = \prod_{i \geq 0} \det\left(1 - \varphi q^{-s} \,\middle|\, H^i_c(\bar{X})\right)^{(-1)^{i+1}}$$

$$= \det\left(1 - \varphi q^{-s} \,\middle|\, H^*_c(\bar{X})\right)^{-1}$$

Let $\mathcal{F}$ be an ℓ-adic sheaf on X. $x \in X$

$$\begin{tikzcd}
\mathrm{Spec}(\overline{k(x)}) \arrow[r] \arrow[rr, bend right, "\bar{x}"'] & \mathrm{Spec}(k(x)) \arrow[r, "x"] & X
\end{tikzcd}$$

$\varphi_x \curvearrowright \mathcal{F}_{\bar{x}} \leftarrow \mathbb{Q}_\ell$-vector space.

automorphism

Def 1:

$$L(\mathcal{F}, s) := \prod_{x \in X} \det(1 - |k(x)|^s \varphi_x \mid \mathcal{F}_{\bar{x}})^{-1}$$

G-L

Def 2:

$$L(\mathcal{F}, s) = \prod_{i \geq 0} \left(\det(1 \mid H^i)\right)^{(-1)^{i+1}}$$

$$\|$$

$$\det(1 - \varphi q^{-s} \mid H^*_c(\bar{X}; \mathcal{F}))^{-1}$$

Let $D^b(X)$ be ~~the~~ bounded derived category of $\mathbb{Q}_\ell$-sheaves on X

$$\mathcal{F} \in D^b(X)$$

for $i \in \mathbb{Z}$ $\quad \mathcal{H}^i(\mathcal{F})$ $\leftarrow$ ℓ-adic sheaf on X.

vanish for almost all i.

<u>DeF:</u>

$$L(\mathcal{F}, s) := \prod_i L(\mathcal{H}^i(\mathcal{F}), s)^{(-1)^i}$$

Version 1

$$L(\mathcal{F}, s) = \prod_{x \in X} \prod_{i \in \mathbb{Z}} \det\left(1 - |k(x)|^{-s} \varphi_x \,\middle|\, H^i(\mathcal{F})_{\bar{x}} \right)^{(-1)^{i+1}}$$

$H^i(\mathcal{F})_{\bar{x}}$.

Version 2:

$$L(\mathcal{F}, s) = \det\left(1 - \varphi q^{-s} \,\middle|\, H^*(\bar{X}; \mathcal{F})\right)^{-1}$$

hypercohomology of $\bar{X}$ w/ coeff. in $\mathcal{F}$.

~~Let X be a variety or~~

Recall: If Y is a geom. connected variety/stack defined over a finite field k,

$$H^1(\overline{Y};\mathbb{Z}/\ell) \cong 0$$

$$H^i(\overline{Y};\mathbb{Z}/\ell) \text{ finite } \forall i$$

Then we have ℓ-adic homotopy groups

$$\pi_n(\overline{Y})_{\mathbb{Q}_\ell} \longleftarrow$$ finite dim'l vector spaces over $\mathbb{Q}_\ell$.

w/ Frobenius

Relative version:

$$Y \xrightarrow[\text{smooth}]{} X \longrightarrow \mathrm{Spec}(\mathbb{F}_q)$$

For $x \in X$, $Y_{\bar{x}} := Y \times_X \mathrm{Spec}(\overline{k(x)})$.

Assume each $Y_{\bar{x}}$ has vanishing $H^1(\ldots, \mathbb{Z}/\ell)$

finite $H^i(\ldots; \mathbb{Z}/\ell)$

vanishing $\pi_n(\ldots)_{\mathbb{Q}_\ell}$ for $n >> 0$.

Claim: $\exists$ $\mathcal{F}_{Y/X} \in D^b(X)$.

with some nice features:

- For each $x \in X$,

$$H^{-i}(\mathcal{F})_{\bar{x}} \simeq \pi_{+i} Y_{\bar{x}}.$$

for all i.

- It is functorial in Y.

$$\begin{tikzcd} Y' \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

gives a map

$$\mathcal{F}_{Y'/X} \to \mathcal{F}_{Y/X} \text{ in } D^b(X).$$

- It is functorial in X

$$\begin{tikzcd} Y' = Y \times_X X' \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

$$\mathcal{F}_{Y'/X'} \simeq f^* \mathcal{F}_{Y/X}.$$

$$G \longrightarrow X$$

group scheme
smooth,
affine,
connected fibers,
generically semisimple + simply connected

alg. curve over $\mathbb{F}_q$

$$BG_X \longrightarrow X$$

Maps $\mathrm{Spec}(R) \to BG_X$

$$\Updownarrow$$

Maps $\mathrm{Spec}(R) \longrightarrow X$
+ G-bundle on $\mathrm{Spec}(R)$.

Get

$$\mathcal{F}_{BG_X/X} \in D^b(X)$$

can build an L-function

$$L(\mathcal{F}_{BG_X/X}, s) =: L(BG_X, s)$$

By definition 1,

$$L(BG_X, s) = \prod_{x \in X} \det\left(1 - |\kappa(x)|^{-s} \varphi_x \,\middle|\, \mathcal{F}_{BG_X/X, \bar{x}}\right)^{-1}$$

$$= \prod_{x \in X} \det\left(1 - |\kappa(x)|^{-s} \varphi_x \,\middle|\, \pi_*(\mathbb{Q}_{BG_X})\right)^{-1}$$

$$L(BG_X, 0) \underset{\text{Steinberg}}{=} \prod_{x \in X} \frac{|BG(\kappa(x))|}{|\kappa(x)|^{\dim BG}}$$

$$= \prod_{x \in X} \frac{|\kappa(x)|^{\dim G}}{|G(\kappa(x))|}.$$

$$L(BG_X, 0) \overset{\text{Def 1}}{\equiv} \begin{array}{l}\text{RHS of} \\ \text{Mass formula} \\ \text{for Weil's conjecture} \\ \text{for } G.\end{array}$$

Relationship between

$$BG_X \longrightarrow X$$

$$\mathrm{Bun}_G(X) \longrightarrow \mathrm{Spec}(\mathbb{F}_q).$$

$$\begin{tikzcd}
X \times \mathrm{Bun}_G(X) \arrow[rr, "\text{restricting to point}"] \arrow[d] \arrow[dr] & & BG_X \arrow[dl] \\
\mathrm{Bun}_G(X) \arrow[dr] & X \arrow[d, "f"] & \\
& \mathrm{Spec}(\mathbb{F}_q) &
\end{tikzcd}$$

Induces a map

$$\mathcal{F}_{X \times \mathrm{Bun}_G(X)/X} \longrightarrow \mathcal{F}_{BG_X/X}$$

$$\text{IS}$$

$$f^* \mathcal{F}_{\mathrm{Bun}_G(X)/\mathrm{Spec}(\mathbb{F}_q)}.$$

Gives
$$\mathcal{F}_{\mathrm{Bun}_G(X)/\mathrm{Spec}(\mathbb{F}_q)} \xrightarrow{\Theta} Rf_* \, \mathcal{F}_{BG_X/X}.$$

<u>Theorem</u>^{A}: (Gaitsgory, L)

Θ is an isomorphism.

<u>Assume this</u>:

Get isomorphisms of $\mathbb{Q}_\ell$ vector spaces

$$\pi_i(\overline{\mathrm{Bun}_G(X)})_{\mathbb{Q}_\ell} \xrightarrow{\sim} H^{-i}(\overline{X}; \mathcal{F}_{BG_X/X}).$$

$$L(BG_X, s) \overset{\text{def 2}}{=} \det\left(1 - q^{-s}\varphi \mid H^*(\bar{X}; \mathcal{F}_{BG/X})\right)^{-1}$$

$$= \det\left(1 - q^{-s}\varphi \mid \pi_* \overline{\mathrm{Bun}_G(X)}_{\mathbb{Q}_\ell}\right)^{-1}$$

$$L(BG_X, s) = \det\left(1 - \varphi \mid \pi_* (\overline{\mathrm{Bun}_G(X)})_{\mathbb{Q}_\ell}\right)$$

$$\overset{\text{Last time}}{=} \frac{|\mathrm{Bun}_G(X)(\mathbb{F}_q)|}{q^{\dim(\mathrm{Bun}_G(X))}}$$

= LHS of Mass formula versia of Weil's conjetur.

Conclusion:

Theorem A $\Longrightarrow$ Weil's conjecture

Note: Theorem A does not reference $\mathbb{F}_q$.

It is "geometric" (can check over $\overline{\mathbb{F}}_q$)

Over $\mathbb{C}$, it follows from the fact that ~~algebraic G bundles on X are~~

$Bun_G(X)$ is also classifier for top. G-bundles on X.

<u>Next time</u>: Sketch proof of Theorem A.

<u>Q</u>: How to define

$$\mathcal{F}_{Y/X} \in D^b(X)$$

for example, if

$$Y = BG_X.$$

<u>Easy case</u>: G is everywhere semisimple

$$H^{-i}(\mathcal{F})_{\bar{x}} \simeq \pi_i(BG_{\bar{x}})_{\mathbb{Q}_\ell}.$$

at generic point, $\pi_i(\overline{BG_M})_{\mathbb{Q}_\ell}$ ← classical

In good reduction case, this rep. is unramified and defines a ℓ-sheaf $\mathcal{F}_i$ on X.

<u>Idea:</u>

Try to define

$$\mathcal{F}_{BG_X/X} \text{ as } \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i[i].$$

<u>Bad definition</u>