III

From HH to Topological HH

$$\text{From HH to } \underbrace{\text{Topological HH}}_{\text{HH over the sphere spectrum}}$$

Categorify HH to adopt highbrow point of view :

$$\begin{tikzcd}
A \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & A^{\otimes 2} \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, shift right=4, leftarrow] & A^{\otimes 3} & \cdots \\
\circlearrowright & \circlearrowright & \circlearrowright & \\
\mathbb{Z}/1 & \mathbb{Z}/2 & \mathbb{Z}/3 & \cdots
\end{tikzcd}$$

<u>Def</u>: A <u>cyclic obj.</u> in a cat. $\mathcal{C}$ is a simplicial obj $X_\bullet$ in $\mathcal{C}$

ie) $X_\bullet : \Delta^{op} \longrightarrow \mathcal{C}$

s.t each $X_n \in \mathcal{C}$ has action by $\mathbb{Z}/n+1$ + axioms.

Eg) $HH(A|k)$ is cyclic obj in k-algs
(has universal property)

Connes' cyclic cat: $\exists$ cat $\Lambda \supseteq \Delta$ s.t

- $\mathrm{ob}\,\Lambda = \mathrm{ob}\,\Delta = \{[n] : n \geq 0\}$
- $\mathrm{Aut}_\Lambda([n]) = \mathbb{Z}/_{n+1}$
- cyclic objs in $\mathcal{C}$ $\longleftrightarrow$ functors $\Lambda^{op} \to \mathcal{C}$

$$\boxed{\text{The circle appears}}$$

Recall: cat $E \rightsquigarrow$ topol. space $|N(E)|$

Fact: $|N(\Lambda)| = BS^1$ is the classifying space for the circle S^1

Consequence: A cyclic obj $X_\cdot$ in k-mods give rise to a "object of $D(k)$ with S^1-action"

$$\underline{X_\cdot} : BS^1 \longrightarrow D(k)$$

simplicial set   dg/∞-cat

Aside: cf, G a finite group.

A functor

$$\underline{Y} : BG \longrightarrow D(k)$$

is data of

- $* \longmapsto$ some complex $Y \in D(k)$
- $g \in G \longmapsto g \circlearrowright Y$

ie) an action of G on Y

ie) Y is a $k[G]$-module

Group cohom:

$$Y^{hG} := \underset{k[G]}{\mathrm{Rhom}}(k, Y) = \lim_{BG} \underline{Y}$$

Group homol:

$$Y_{hG} := Y \overset{\mathbb{L}}{\underset{k[G]}{\otimes}} k = \operatorname{colim}_{BG} \underline{Y}$$

Tate cohom:

$$Y^{tG} := \underbrace{\mathrm{hofib}}_{\text{cone}}\left(Y_{hG} \xrightarrow[g \in G]{\mathrm{Norm} = \sum g} Y^{hG} \right)$$

Back to S^1: $\underline{X} : BS^1 \to D(k)$ is the data of

- $\underbrace{\text{chain complex}}_{\text{Dold-Kan of } X.}$ $X \in D(k)$
- module structure over
$$k[S^1] := k[B\mathbb{Z}] \simeq k[\varepsilon]/\varepsilon^2$$

where ε is in homol deg 1

ie) $\varepsilon : X \longrightarrow X[-1]$

$\left(\overset{\shortparallel}{B} \right)$

Mimik earlier def^{n}s:

$$X^{hS'} := R\mathrm{hom}_{k[S']}(k, X) = \lim_{BS'} \underline{X}.$$

$$X_{hS'} := k \overset{L}{\otimes}_{k[S']} X = \operatorname{colim}_{BS'} \underline{X}.$$

$$X^{tS'} := \operatorname{hofib}\left(X_{hS'}[1] \xrightarrow{\text{Norm}} X^{hS'} \right)$$

$\rotatebox{90}{$\in$}$

$D(k)$

If $X_\cdot = HH(A|k)$, then

$$HH(A|k)^{hS^1} \simeq HC^-(A|k)$$

$$HH(A|k)_{hS^1} \simeq HC(A|k)$$

$$HH(A|k)^{tS^1} \simeq HP(A|k)$$

May now replace $A \in k\text{-alg} \subseteq D(k)$ by any alg. obj. in any rich enough symmetric monoidal derived / ∞-cat

eg) spectra Sp.

$$D(k) \xrightarrow[\mathbb{S} \to k]{\text{restriction along}} Sp = D(\mathbb{S})$$

- Sym. monoidal stable ∞-cat
- unit $= k$

- Sym. monoidal stable ∞-cat
- unit $= \mathbb{S}$ sphere spectrum

A alg. obj. $\rightsquigarrow A$ alg obj

$\rightsquigarrow$ cyclic obj $HH(A/\mathbb{S})$

$\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad THH(A) \in Sp$

topological HH

$THH(A)^{hS^1} =: TC^-(A)$ negative top. cyclic hom.

$THH(A)_{hS^1} \neq:$ top. cyclic homology

$THH(A)^{tS^1} =: TP(A)$ periodic top. cyclic hom.

$\in Sp$

$\rightsquigarrow$ homotopy groups $THH_n(A)$, $TC^-_n(A)$, $\pi_n\left(THH(A)_{hS^1}\right)$, $TP_n(A)$

$THH_n(A)$: A-modules

$TC^-_n(A)$, $\pi_n\left(THH(A)_{hS^1}\right)$, $TP_n(A)$: $\mathbb{Z}$-modules

Comparison with HH:

$$THH_n(A) \longrightarrow HH_n(A)$$

kernel + cokernel killed
by some $N = N(n)$

$\Rightarrow$ If $A \supseteq \mathbb{Q}$ then it is an isom.

But very interesting if A is
char p or p-adic.