End of $\mathbb{F}_p$ story

<u>Thm</u> (Bhatt - M. - Scholze) : R smooth $/\mathbb{F}_p$.

Then $TP(R)$ has a filtration with graded pieces

$$\underbrace{R\Gamma_{crys}(R/\mathbb{F}_p)}_{= \Omega^{\bullet}_{\tilde{R}/\mathbb{Z}_p}}[2i] \qquad (i \in \mathbb{Z})$$

$\tilde{R}$ smooth lift of R to $\mathbb{Z}_p$

<u>Overview</u>

$$TP(R) \xrightarrow{\sim} \mathrm{Tot}\big(TP(R_{perf}) \rightrightarrows TP(R_{perf} \otimes_R R_{perf}) \rightrightarrows$$

Sunday

$\Rightarrow$ reduces to analysis

$TP_0(A)$ $\quad$ A quasiregular semiperfect $= B/I$

- B perfect
- $I \subseteq B$ reg. seq.

- p-torsion free complete filtered
- graded $= THH_{2*}(A) \xleftrightarrow{\text{this morning}} HH_{2*}(A) = \Gamma^*_A(I$

· $TP_0(A)/p = HP_0(A/\mathbb{F}_p)$

$=$ divided power envelope

Sem. of $B \twoheadrightarrow A$

· $W(B)$-alg

<u>Key</u>: $TP_0(A) := \underbrace{\hat{A}_{crys}(A)}_{\text{crystalline period ring}}$

$:=$ divided power envelope of

$$W(B) \twoheadrightarrow B \twoheadrightarrow A$$

<u>Key case</u>: $A = \mathbb{F}_p[t^{\pm 1/p^\infty}]/t-1$

$$= \mathbb{S}[\mathbb{Q}_p/\mathbb{Z}_p] \otimes_{\mathbb{S}} \mathbb{F}_p$$

§ THH etc of $\mathbb{Z}_p$-algs.

Key starting point this morning

$$THH_*(\mathbb{F}_p) \cong \mathbb{F}_p[u]$$

$(u \in THH_2(\mathbb{F}_p))$

Qu: Given a ring A, when is it true that

$$THH_*(A) \cong A[u]$$

$(u \in THH_2(A))$?

- $A = \mathbb{F}_p$ ✓
- A is any perfect $\mathbb{F}_p$-alg ✓
- A is any perfectoid ring ✓

(· ?)

mixed char. analogues of perfect $\mathbb{F}_p$-algs.

Def^n: A ring A is per<u>fect</u>oid if

(1) it is p-adic complete and p-torsion free

(2) $A/p \longrightarrow A/p$, $a \mapsto a^p$ is surj.

(3) $\exists\, u \in A^\times$ s.t pu has a p^{th} root

(4) A is p-closed in $A[\frac{1}{p}]$

Eg) (i) $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p} \supseteq \mathcal{O}$ its ring of integers.

perfectoid

(ii) A perf'd $\Rightarrow A\langle T^{1/p^\infty}\rangle$ focus on this case.

Thm (Hesselholt) : $THH_*(\mathcal{O}) \cong \mathcal{O}[u]$.

Idea:

$$\begin{array}{ccccc} THH_*(\overline{\mathbb{F}}_p) & \longleftarrow & THH_*(\mathcal{O}) & \longrightarrow & HH_*(\mathcal{O}/\mathbb{Z}_p)[\tfrac{1}{p}] \\ \checkmark & \text{kill } p^{1/p^\infty} & & \text{invert } p & \checkmark \end{array}$$

$\square$

$TP(\mathcal{O})$?

$$\mathbb{F}_p \overset{TP_0}{\rightsquigarrow} \mathbb{Z}_p$$

$$\mathcal{O} \underset{TP_0}{\rightsquigarrow} ??$$

Aside on p-adic Hodge Thy:

$\mathcal{O} \rightsquigarrow$ its <u>tilt</u> $\mathcal{O}^\flat := (\mathcal{O}/p\mathcal{O})^{perf}$

$$= \lim_{\substack{\longleftarrow \\ x \mapsto x^p}} \mathcal{O}/p\mathcal{O}$$

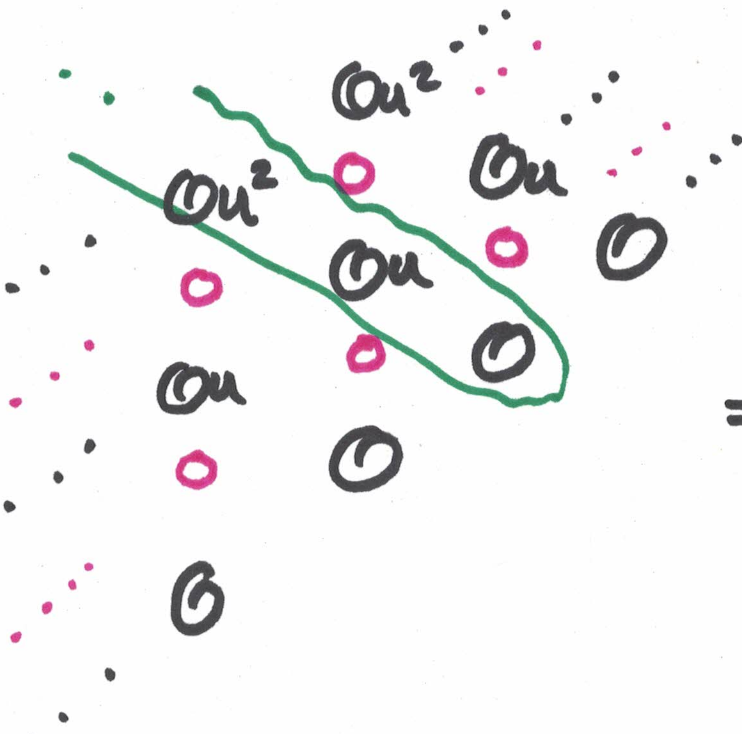
$$= \{(x_0, x_1, \dots) : x_i \in \mathcal{O}/p,\ x_i^p = x_{i-1}\}$$

perfect $\mathbb{F}_p$-alg $\qquad \ni \ (p, p^{1/p}, p^{1/p^2}, \dots) = p^\flat$

Witt vectors

$$\begin{tikzcd} & W(\mathcal{O}^\flat) =: A_{inf} \arrow[dl, "\bmod p"'] \arrow[dr, "\bmod \underbrace{p-[p^\flat]}_{\xi}"] & \\ \mathcal{O}^\flat & & \mathcal{O} \end{tikzcd}$$

Fontaine's infinitesimal period ring



$\Rightarrow TP_*(\mathcal{O})$

$\Rightarrow TP_0(\mathcal{O})$ = either $\mathcal{O}[[u]]$

or A_{inf} ← it's this one.

Consequence: For any $\mathcal{O}$-alg A, then

$$\underbrace{TP(A)}/\xi \simeq HP(A|\mathcal{O})$$

related to some lift to $A_{\inf}$ of the de Rham cohom of A.

rel'n to de Rham cohom.

Thm (Bhatt–M.–Scholze). There exists a cohom theory

$$\text{smooth } \mathcal{O}\text{-alg } R \longmapsto \text{complex of } A_{\inf}\text{-modules } \Delta_{R/\mathcal{O}} \left(= A\Omega_R \text{ old-fash.}\right)$$

with properties:

① It lifts de Rham cohom

$$\Delta_{R/\mathcal{O}} / \xi \simeq \Omega^{\bullet}_{R/\mathcal{O}}$$

② $TP(R)$ has a filtr. with graded pieces

$$\Delta_{R/\mathcal{O}}[2i] \qquad (i \in \mathbb{Z})$$

③ $\Delta_{R/\mathcal{O}} \longleftrightarrow \mathbb{Z}_p$-étale cohom of $R[\frac{1}{p}] / \mathbb{C}_p$

Key words:

$$TP(R) \xrightarrow{\simeq} \mathrm{Tot}\Big(TP(R_{perfd}) \rightrightarrows TP(R_{perfd} \otimes_R R_{perfd}) \cdots$$

$TP_0\left(\begin{matrix}\text{quasi regular}\\ \text{semiperfectoid}\end{matrix}\right)$ = prismatic period ring

prismatic cohom
Bhatt–Scholze
(replace divided powers ..

.. by p-derivations δ)

$\rightsquigarrow$ $TP(R) \longleftrightarrow$ prismatic cohom

$\Delta_{R/G}$

$\square$