Correction: $\partial^{\pi}_{v} \eta = \eta \, \partial^{\pi}_{v}$ (defines ∂^{π}_{v})

$\partial^{\pi}_{v} \eta = 0$

Sheaves: K^{MW}_{*}, GW

$GW(\operatorname{Spec} L \longrightarrow \operatorname{Spec} K) =$ restriction or $\otimes_K L$ of bilinear forms

$GW(K) \longrightarrow GW(L)$

<u>Transfers</u>: $K \subseteq L$ finite extension of finite type schemes $/ \mathbb{R}$

$$Tr_{L/K} : GW(L) \longrightarrow GW(K)$$

geometric transfer, cohomological transfer, absolute transfer

depends on generators

doesn't

twisted

$$Tr_{L/K}(B : V \times V \to L) =$$

$$V \times V \xrightarrow{B} L \xrightarrow{Tr_{L/K}} K$$

when $K \subseteq L$ is separable

geometric: $L = K[z]/\langle f \rangle$

$$\operatorname{Spec} L \overset{z}{\hookrightarrow} \mathbb{P}^1_K$$

$$\mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K / \mathbb{P}^1 - \{z\} \simeq \mathbb{P}^1_L$$

$$\cancel{}\, K^{MW}_1 (\mathbb{P}^1_K \longrightarrow \mathbb{P}^1_L) \text{ is a map}$$

$$\mathrm{Tr}^{geom}_{L/k} : GW(L) \longrightarrow GW(K)$$

<u>CH</u> Chow groups $X \in Sm_k$

$X^{(i)}$ = codim i reduced, irreducible subschemes of X

$$CH^i(X) = \bigoplus_{X^{(i)}} \mathbb{Z} \Big/ \text{rational equivalence}$$

$$V \subset X \times \mathbb{P}^1$$

$$V \cap \{X \times \{0\}\} \sim V \cap (X \times \{\infty\})$$

useful in enum geom: Chern classes, push forwards, pullbacks ring structure

Bloch $CH^i(X) = H^i(X, K^M_i)$

Oriented Chow groups or Chow-Witt groups

$$\widetilde{CH}^i(X) = H^i(X, K_i^{MW})$$

elts are formal combinations $Z \in X^{(i)}$ and $BGGW(k(z))$

Barge-Morel

computed by Rost-Schmidt complex

$$\bigoplus_{z\in X^{(i-1)}} K_1^{MW}(k(z), \det) \to \bigoplus_{z\in X^{(i)}} GW(k(z), \det_{k(z)} T_z X) \to \bigoplus_{z\in X^{(i+1)}} K_{-1}^{MW}(k(z), d$$

Fasel   pullbacks $f: X \to Y$

M. Levine   pushforwards

non-commutative ring structure

~~[illegible]~~

E field

Λ 1-dim'l E vector space

$$K_i^{MW}(E, \Lambda) = K_i^{MW}(E) \otimes_{\mathbb{Z}[E^*]} \mathbb{Z}[\Lambda - 0]$$

$$\rightsquigarrow \widetilde{CH}^i(X, L) = H^i(X, K_i^{MW}(L$$

$L \to X$ line bundle

$f: X \to Y$ proper $\dim Y - \dim X = r$

$$f_* : \widetilde{CH}^i(X, \omega_{X/k} \otimes f^*\mathcal{L}) \to \widetilde{CH}^{i-r}(Y, \omega_{Y/k} \otimes \mathcal{L})$$

$\det TX$

Lecture 3: Degree via local degree

Alg top: $F: S^n \to S^n$ $p \in S^n$

$F^{-1}(p) = \{q_1, \ldots, q_n\}$

$$\deg F = \sum \deg_{q_i} F$$

Differential topology formula for $\deg_{x_i} F$:

Choose coords $x_1, \ldots, x_n$ near q_i

$y_1, \ldots, y_n$ near p

$$F: \mathbb{R}^n \to \mathbb{R}^n$$

$$\mathrm{Jac}\, F = \det \frac{\partial F_i}{\partial x_i}$$

$$\deg_{q_i} f = \begin{cases} 1 & \text{if } \mathrm{Jac}\, f > 0 \\ -1 & \text{if } \mathrm{Jac}\, f < 0 \end{cases}$$

Q: what if $\mathrm{Jac}\, f = 0$

$\mathbb{A}^1$-alg top

Lannes / Morel: $f: \mathbb{P}^1 \to \mathbb{P}^1 \ / k$

$p \in \mathbb{P}^1(k)$

$f^{-1}(p) = \{q_1, \ldots, q_n\}$

$$\deg^{\mathbb{A}^1} f = \sum \langle \mathrm{Jac}_{q_i} f \rangle \in GW(k)$$

this doesn't depend on p!

<u>Prop</u>: (Global degree is a sum of local degrees)

$F: \mathbb{P}^n \to \mathbb{P}^n$ finite $f^{-1}(\mathbb{A}^n) = \mathbb{A}^n$

$$\mathbb{P}^n/\mathbb{P}^{n-1} \xrightarrow{\bar{F}} \mathbb{P}^n/\mathbb{P}^{n-1}$$

$$\deg^{\mathbb{A}^1} F = \sum_{q \in F^{-1}(p)} \deg_q^{\mathbb{A}^1} F$$

$$p \in \mathbb{A}^n(k)$$

where $\deg_q^{\mathbb{A}^1} F$ is degree of composite

$$\begin{tikzcd}
\mathbb{P}^n/\mathbb{P}^{n-1} \arrow[r, phantom, "\simeq"] & U/(U-q) \arrow[r] & \mathbb{A}^n/(\mathbb{A}^n - p) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \mathbb{P}^n/\mathbb{P}^{n-1} \\
\text{if } k(q) = k & & Th(N_p \mathbb{A}^n) & \\
\text{otherwise } \mathbb{P}^n/\mathbb{P}^{n-1} \arrow[r] \arrow[rrruu, bend right] & \mathbb{P}^n/(\mathbb{P}^n - q) \arrow[uu] & &
\end{tikzcd}$$

- If f is étale at q, then $\deg_q^{\mathbb{A}^1} f = \mathrm{Tr}_{k(q)/k} \langle \mathrm{Jac} f(q)$ and $k(q) \supseteq k$ separable

A: Eisenbud-Levine-Khimshiashvili Signature formula

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$0 \longmapsto 0 \quad \text{isolated zero}$$

$$\deg_0 f = \text{signature } \omega^{EKL}$$

ω^{EKL} is a bilinear form on

$$Q = \frac{\mathbb{R}[x_1, \ldots, x_n]_0}{\langle f_1, \ldots, f_n \rangle}$$

char $\nmid$ dim Q

$$\operatorname{Jac} f \in Q$$

pick any $\eta: Q \longrightarrow \mathbb{R}$ $\mathbb{R}$-linear s.t.

$$\eta(\operatorname{Jac} f) = \dim Q$$

otherwise use a distinguished socle element in place of Jac

$$\omega^{EKL}: Q \times Q \longrightarrow \mathbb{R}$$
$$(a,b) \longmapsto \eta(ab)$$

179

<u>Q</u> (Eisenbud): ω^{EKL} could be a degree, even replacing $\mathbb{R}$ with k.

Does this have an interpretation?

<u>Thm</u> (Kass - W.)

$$\deg_0^{\mathbb{A}^1} f = \omega^{EKL}$$

project: remove $k(x) = k$ hypothesis

<u>Ex</u>: ω^{EKL} for $f(x) = x^2$

$$Q = k[x]/\langle x^2 \rangle \qquad \text{basis } \{1, x\}$$

$$\operatorname{Jac} f = 2x$$

$$\eta : k[x]/\langle x^2 \rangle \longrightarrow k$$

$$\eta(2x) = 2$$
$$\eta(1) = 0$$

$$\begin{array}{c|cc} & 1 & x \\ \hline 1 & 0 & 1 \\ x & 1 & 0 \end{array} \qquad \omega^{EKL} = \langle 1 \rangle + \langle -1 \rangle$$

$\mathbb{A}^1$ - Milnor numbers

joint with Jesse Kass

<u>Def</u>: A point p on a scheme X is a node if after base change to k^s

$$\widehat{\mathcal{O}}_{X,p} \cong \frac{k^s[[x_1,\ldots,x_n]]}{x_1^2+\ldots+x_n^2+\text{higher order terms}}$$

Let X be a hypersurface $X = \{f=0\} \subseteq \mathbb{A}^n$

$p \in X$ be a singularity

- As X is perturbed in a family, p bifurcates into nodes
- For $(a_1,\ldots,a_n)$ have a family of hypersurfaces $f(x_1,\ldots,x_n) + a_1x_1 + a_2x_2 + \ldots + a_nx_n = t$ parametrized by t

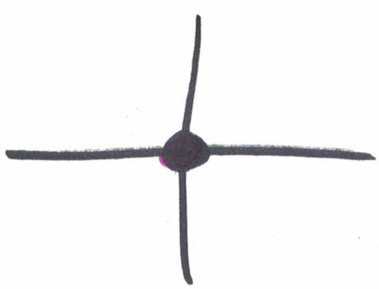
Milnor

$k = \mathbb{C}$ For any sufficiently small $(a_1, \ldots, a_n)$ the family contains $\mu(p)$ nodes

$$\mu(p) = \text{Milnor } \#$$
$$= \deg(\operatorname{grad} F)(p)$$

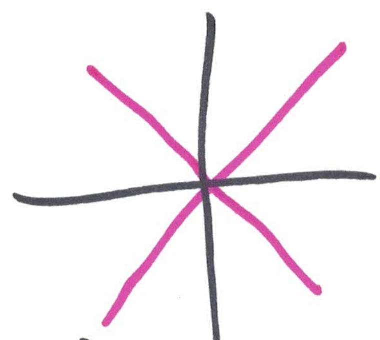
When k is not alg closed, nodes contain arithmetic data

$k = \mathbb{R}$



$x^2 + y^2 = 0$

non-split node, i.e. tangent directions not defined over k



$x^2 - y^2 = 0$

split node

$\underline{\text{Def}}$: The $\underline{\text{type}}$ of a node $p \in \{f=0\}$

$$\mathrm{type}(p) = \deg_p^{\mathbb{A}^1} \operatorname{grad} F \in GW(k)$$

$\underline{\text{Ex}}$:

~~Equivalently~~, ~~we~~ choose preimage of p after base change to $k(p)$.

$$\widehat{\mathcal{O}}_{X,p} = \frac{k(p)[[x_1,\ldots,x_n]]}{a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + \text{higher order terms}}$$

$$\mathrm{type}(p) = \mathrm{Tr}_{k(p)/k} \langle 2^n a_1 a_2 \cdots a_n \rangle$$

always separable extension

Ex $\quad \operatorname{type}(x^2 + a y^2) = \langle a \rangle$

Def: p hypersurface singularity

$$p \in \{ f = 0 \}$$

$$\mu^{\mathbb{A}^1}(p) = \deg_p \operatorname{grad} f$$

Thm: (Kass-W.) For generic $(a_1, \ldots, a_n)$

$$\sum_{\substack{x \text{ nodes} \\ \text{in family}}} \operatorname{type}(x) = \mu^{\mathbb{A}^1}(p)$$

in $GW(k)$

$\underline{\text{Ex}}$: $F(x,y) = y^2 - x^3$ char $k \neq 2, 3$

$\text{grad } F = (-3x^2, 2y)$

- $\mu^{\mathbb{A}^1}(0) = \deg_0 \text{grad } F$

$$= \deg_0 (x \mapsto -3x^2) \deg_0 (y \mapsto 2y)$$

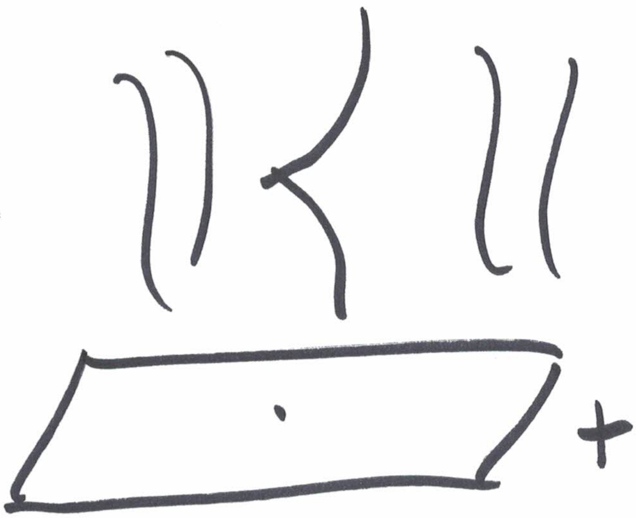
$$= \langle -3 \rangle (\langle 1 \rangle + \langle -1 \rangle) \langle 2 \rangle$$

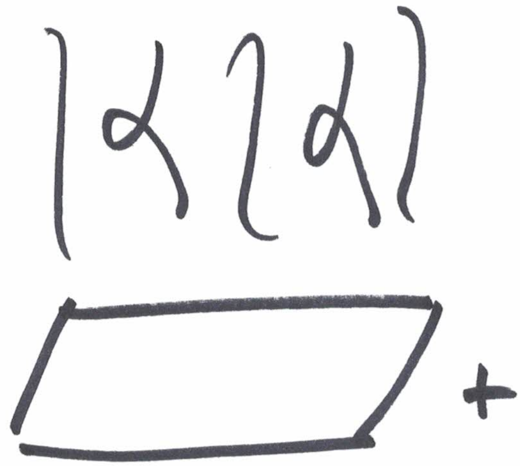
$$= \langle -6 \rangle + \langle 6 \rangle = \underbrace{\langle 1 \rangle + \langle -1 \rangle}_{= h}$$

- Family parametrized by t

$$y^2 = x^3 + ax + t$$

$a = 0$ $a \neq 0$





nodes occur when
$x^3 + ax + t$ has
a double root

$\Longleftrightarrow$

$-27t^2 - 4a^3$

$\mathbb{F}_5$: $\langle 1 \rangle = \langle -1 \rangle$

in family can't
have one split
and one non-split
rat'l node

$\mathbb{F}_7$: $\langle 1 \rangle \neq \langle -1 \rangle$

can't have 2 split
or 2 non-split rat'l nodes