Recall: 
\[
\begin{align*}
\frac{1}{h(D_1, D_2)} &= \sum_v h_v(D_1, D_2) \\
&= \underbrace{\int_{D_1} w_{D_1}}_{h_p(D_1, D_2)} + \sum_{v \neq p} h_v(D, D_2)
\end{align*}
\]

Constructed \( w_p \), (3rd kind diffs)

How do we compute Coleman integrals of diffs of 3rd kind? \( \int_S^R w, \text{Res}(w) = (P) - (Q) \).

1) Compute \( \mathcal{R}(w) \in H^1_{\text{dR}}(X) \) by computing
   cup products

   \[ \mathcal{R}(w) = \sum b_i w_i \quad \text{for } \{ w_i \} \text{ basis of } H^1_{\text{dR}}(X) \]

   by computing \( \mathcal{R}(w) \cup [w_j] \)

2) Let \( d := \phi^* w - p \cdot w \). Use Frob. equivariance to compute
   \[ \mathcal{R}(x) = \phi^* \mathcal{R}(w) - p \cdot \mathcal{R}(w) \]

3) Let \( \beta \) be s.t. \( \text{Res}(\beta) = (R) - (S) \). Compute \( \mathcal{R}(\beta) \).
4) Using Coleman reciprocity:

\[ \int_{S}^{R} w = \frac{1}{1-p} \left( \Xi(a) \cup \Xi(b) + \sum_{A \in \mathbb{X}(Q/p)} \text{Res}_{A} (dS_{\beta}) \right) \\
- \left( \int_{\phi(s)}^{\phi(r)} w - \int_{R}^{\phi(r)} w \right) \]

This lets us compute \( h_{p}(D_{1}, D_{2}) \), since

\[ h_{p}(D_{1}, D_{2}) = \int_{D_{2}} w_{D_{1}}. \]

What about the self-pairing of a divisor?

\[ h_{p}(D, D) ?? \]

It turns out that if we consider the case of \( X/Q \) a hyperelliptic curve of odd degree model, genus \( g \)

\[ h_{p}(D, D) = -2 \sum_{t=0}^{g-1} \omega_{i} \bar{\omega}_{i} \]

when \( D = (z) - (\infty) \)

\[ \omega_{i} = \frac{x^{i}dx}{2y} \]
\[ \bar{\omega}_{i} \text{ dual under } \mathcal{U} \]

Can use this to study integral pts on hyperelliptic curves.
Quadratic Chabauty for integral
pts on hyperell. curves (B-Besser-Müller)

let $f \in \mathbb{Z}[x]$, monic, separable, $\deg 2g+1 \geq 3$

let $U = \text{Spec} \left( \mathbb{Z}[x,y]/(y^2-f(x)) \right)$, $X$ be normalization
of proj. closure of generic fiber of $U$.

let $J$ be the Jacobian of $X$. Assume
$\text{rk } J(\mathbb{Q}) = g$, suppose $\text{log: } J(\mathbb{Q}) \otimes \mathbb{Q}_p \to H^0(X_{\mathbb{Q}_p}, \mathcal{O})$
is an isomorphism.

let $p$ be a good prime.

Then $\exists \ a_{ij} \in \mathbb{Q}_p$ s.t.

$$ p(z) = -2 \sum_{i=0}^{g-1} \int_0^z w_i \overline{w_i} - \sum_{i,j} a_{ij} \int_{\infty}^z w_i \int_{\infty}^z w_j,$$

$$w_i = \frac{x^i \text{d}x}{2y}$$

takes values in an explicitly computable finite set $S \subseteq \mathbb{Q}_p$
for all $z \in U(\mathbb{Q})$. 
Idea: \( h = h_p + \sum_{l \neq p} h_l \)

\[ h - h_p = \sum_{l \neq p} h_l \]

Coleman integrals on integral pts, finitely many values, can compute them at the start.

\[ \sum_{i,j} S_{i,j} S_{i,j} = \text{solve for } a_{i,j} \]

What goes wrong for rational points?

- We don't know how to control \( \sum_{l \neq p} h_l \) on all rational points.

Goal: Extend QC from integral points to rational points

Problem: Need to control local heights away from \( p \).
Solution: Use height that factors through Kim's unipotent Kummer map, can control local heights in this setting.

- For this, need "non-abelian" height (instead of heights via the Jacobian)
- Use heights on Bloch-Kato Selmer groups

Let $X/\mathbb{Q}$ be a nice curve $g \geq 1$, $p$ good prime.

Let $V = H^1_{et}(X_{\bar{\mathbb{Q}}})^*$

By Nekovář (1993), have a bilinear symmetric pairing

$$h: H^1_f(G_{\mathbb{Q}}, V) \times H^1_f(G_{\mathbb{Q}}, V^*(1)) \to \mathbb{Q}_p$$

where $h = \sum h_v$

This is equivalent to the Coleman-Gross height via an étale Abel-Jacobi map (Besser)

This height $h$ also depends on choices, like the C-G height:
1) the choice of an idèle class char
\[ \chi: \mathbb{A}^x_\mathbb{Q} / \mathbb{Q}^x \to \mathbb{Q}_p \]

2) a splitting of the Hodge filtration
\[ \text{on } V_{deR} = \text{Dcris}(V) = H^1_{deR}(X_{\mathbb{Q}_p})^* \]

Recall that in the Coleman-Gross height, to pair points on the Jacobian, needed choice of divisors. Here depends on mixed extensions:

Given a pair of extn classes
\[ (c_1, c_2) \in H^1_f(\mathcal{E}_a, V)^* \times H^1_f(\mathcal{E}_a, V^*(1)) \]

take reps \( E_1, E_2 \):

\[ \begin{array}{cccc}
0 & \to & \mathcal{O}_a & \to \mathcal{E}_2 & \to \mathcal{O}_p & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & V & \to E_1 & \to \mathcal{O}_p & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
& & Q_p & & & \\
\end{array} \]

fill in diagram
$E$ is a mixed extn of $E_2$ with graded pieces $Q_p$, $V$, $Q_p(1)$.

with a weight filtration

$0 = W_3 E E W_2 E E W_1 E E W_0 E = E$ s.t.

$W_{-1} E = E_2 \quad W_0 E / W_2 E = E_{-1}$
Let $\mathcal{M}_a = \frac{1}{2}$ such mixed extensions.

$\nu$ prime $\Rightarrow \mathcal{M}_r = \{ \text{mixed extns of } G_v\text{-reps} \}$

$\mathcal{M}_a, f$ : crystalline

For $E \in \mathcal{M}_a, f$, define $h_{\nu}(E) = h_{\nu}(\text{loc}_v, E)$

and then define

$$h(e_1, e_2) = \sum_{\nu} h_{\nu}(E)$$

From this point forward, we'll assume that $h_{\nu}(E_2) = 0 \text{ for } \nu \neq \rho$

(e.g. when $X$ has potential good reduction at $\ell$, local height $h_{\ell} = 0$).
Def. A filtered φ-module (over Qp) is a finite-dimensional Qp-vector space W, with an exhaustive and separated decreasing filtration Fil i and an automorphism φ:
- exhaustive: W = ⋃ Fil i
- separated: \( \cap_i \text{Fil}_i = 0 \)
- decreasing: Fil i+1 ⊆ Fil i

Examples:
1) Qp with Fil 0 = Qp, Fil n = 0 for all n > 0, φ = id

2) By Faltings' companion theorem, have \( H^1_{\text{dR}}(X_{\overline{\mathbb{Q}_p}}) = \text{Dens}(H^1_{\text{et}}(X_{\overline{\mathbb{Q}}}, Q_p)) \) and \( H^1_{\text{et}}(X_{\overline{\mathbb{Q}}}, Q_p) \) is crystalline, take Frobenius \( \phi \) on crystalline cohomology and Hodge filtration \( \Rightarrow H^1_{\text{dR}}(X_{\overline{\mathbb{Q}_p}}) \), the structure of a filtered φ-module.

3) \( \text{VdR} = H^1_{\text{dR}}(X_{\overline{\mathbb{Q}_p}})^* = \text{Dens}(V) \) with dual filtration and action.
4) The direct sum $\mathcal{O}_p \oplus \mathcal{O}_p \oplus \mathcal{O}_p(1)$ has the structure of a filtered $\phi$-module as well.

Let $E_p \in \mathcal{M}_{p,f}$.

Then $E_{Dr} = \text{Dens}(E_p)$ is a mixed-extn of filtered $\phi$-modules w/ graded pieces $\mathcal{O}_p, V, \mathcal{O}_p(1)$.

To construct the local height $h_p$ of $E_p$, need an explicit description of

- Frobenius $\phi$
- filtration on $E_{Dr}$

Want to compute $h(E) = \sum h_v(E_v)$

$$= h_p(E_p) + \sum_{v \neq p} h_v(E_v)$$

From Kim to Nekovář:

Idea: Want maps: $X(\mathbb{Q}) \to M_{Q,f}$

$X(\mathbb{Q}_p) \to N_{p,f}$

$X(\mathbb{Q}_\ell) \to M_{\ell}$

factor through unipotent Kummer map
Assume in addition to $X/\mathbb{Q}$ with $g \geq 2$, \$11
rk $J(\mathbb{Q}) = g$, \$4 \text{ rk NS } (J) > 1$

Then $\exists \tilde{z} \in \text{Prc } (X \times X)$ that allows us to \$4-5
construct a nice quotient of $U_2$ \$11 \text{ in notes}
(by Kim: $U_n = n$-unipotent quotient of \$4-5
$\pi^{et} (X_{\mathbb{Q}})$)

By work of Kim, have local unipotent Kummer maps \$11
$j_{u,v} : X(\mathbb{Q}_v) \to H^1 (G_v, U)$

We'll assume that $j_{u,v}$ is trivial for all \$11
$l \neq p$
(in general, by Kim-Tamagawa, know that \$11
$j_{u,l}$ has finite image)
* this assumption is satisfied in the case \$11
of $X$ having everywhere pot. good red.
So we have the following diagram:

\[ X(Q) \rightarrow X(Q_p) \]

\[ \downarrow j_u = : j \downarrow i_{u,p} = : j_p \]

\[ H^1_f(G_T, U) \rightarrow H^1_f(G_p, U) \]

**Lemma.**

The set

\[ X(Q_p)u = j_p^{-1}(\text{loc}_p(H^1_f(G_T, U))) \text{ is finite}. \]

More generally, this result holds for \( r < g + rk\text{NS}(T)-1 \).

We have \( X(Q) \subseteq X(Q_p)u \) and the goal is to compute \( X(Q_p)u \) using p-adic heights "quadratic Chabauty" for rat'll points.