Our set-up:
$X / \mathbb{Q}$ nice curve of genus $g \geq 2$
$\operatorname{rk} J(\mathbb{Q}) = g$, $\operatorname{rk} \operatorname{NS}(J) > 1$, $\log : J(\mathbb{Q}) @ \mathbb{Q}_p \to \mathbb{H}^2(X, \mathbb{Z})$
$p$ good prime
$X$ has everywhere potent. good reduction.

We extend yesterday's diagram:

$$
\begin{array}{c}
X(\mathbb{Q}) \xrightarrow{j} X(\mathbb{Q}_p) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H^1_f(G_\mathbb{Q}, U) \xrightarrow{\text{loc}_p} H^1_f(G_p, U) \to U^\text{dR}/\text{Fil}^0 \\
\downarrow \tau \quad \downarrow \tau \quad \downarrow \quad \downarrow \\
M_{\mathbb{Q}, f} \to M_p \to M_{\text{fil}, t} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H^1_f(G_\mathbb{Q}, V) \times H^1_f(G_\mathbb{Q}, V^{+2}) \xrightarrow{h_p} \\
\downarrow \quad \downarrow \\
\mathbb{Q}_p \xrightarrow{h} \\
\end{array}
$$

Construct $\tau$, $\tau_p$ via twisting.
Then by our assumptions, have the following:

\[ \text{Nekovar} \quad \text{ht} \quad : \quad H^0_f(G_t, V) \times H^0_f(G_t, V^*(1)) \]

\[ \text{loc}_p, \text{ Poincare duality} \]

\[ H^0_f(G_p, V) \times H^0_f(G_p, V) \]

\[ \text{Blasch-Kato} \]

\[ H^0(X, \Omega^{'})^* \times H^0(X, \Omega^{'})^* \]

so we may view Nekovar height as a bilinear pairing.

Now let \( A : X(Q) \rightarrow M_{\alpha, f} \)

\[ x \mapsto \tau(j(x)) \]

and do this similarly for \( x \in X(Q_p) \)

\[ \Rightarrow x \mapsto h(\tau(j(x)) \text{ extends to} \]

\[ X(Q_p) \rightarrow Q_p \]

Fix a basis \( \{ \psi_i \} \) of \( H^0(X, \Omega^{'})^* \otimes H^0(X, \Omega^{'})^* \)

rewrite \( \text{ht} \) in terms of this basis, using known \( Q \)-points (either \( T(Q) \) or \( J(Q) \)).
The function \( p: X(\mathbb{Q}_p) \to \mathbb{Q}_p \)
\[ x \mapsto h_p(A(x)) - h(A(x)) \]
vanishes on \( X(\mathbb{Q}_p) \) and has finitely many zeros.

To make this explicit, one need to:
1) Write \( h \) in terms of basis of \( H^0(X, \Omega^1)^* \otimes H^1(X, \mathbb{Q})^* \)
2) Compute \( h_p \circ A \to \) using filtered \( \phi \)-module structure of \( \text{Der}(A(x)) \).

**Lemma.** There exists a connection \( A_2 \) with Hodge filtration and Frobenius structure s.t.
\[ x^* A_2 = \text{Der}(A(x)) \]
(This follows from Olsson's companion theorem.)

\\( A_2 \) is a unipotent isocrystal, quotient of universal 2-step unipotent conn. \( A_2^{dr} \).

Suffices to compute:
1) Hodge filtration
2) Frobenius structure.
1) Hodge: defined by Hodge filtration on graded pieces and its global nature
(Hadrian; universal properties)

2) Frobenius: via Frob on $A_{rig}$ and comparison
Thm of Chiarellotto-Le Stum; initial
condition $n \mapsto$ gives a $p$-adic differential
equation that we solve using Tuitman's algorithm

(§§ 5.2-5.3 in notes for more details)

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Examples of Quadratic Chabauty

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A problem of Diophantus (Problem 17, Book VI of Arithmetica):

Find three squares which when added give a square and s.t. the first one is the square root of the second and the second is the square root of the third: i.e., can one find positive, rational, $x, y$ s.t. $y^2 = x^8 + x^4 + x^2$?
Diophantus found $x = \frac{1}{2}, y = \frac{1}{16}$. Are there any others?

Remove the singularity at $(0,0) \rightarrow$ want $X(Q)$ for $X : y^2 = x^6 + x^2 + 1$.

$J(Q)$ has $rk \ 2$

$J \sim E_1 \times E_2, \ rk \ NS(J) = 2$.

Wetherell ('97) : determined $X(Q)$ via covering collections and classical Chabauty-Coleman

Bianchi ('19) gave a Q.C.-solution to Diophantus' question using $p$-adic sigma function $X(Q) = \{ oo, (0, \pm 1), (\pm 1/2, \pm 9/8) \}$.

B. Dogra ('16) : can apply QC to bielliptic genus 2 curves $X/K$ ($K = \mathbb{Q}$ or quadratic imag.) with $rk J(K) = 2$ (computational tools: $p$-adic heights on elliptic curves, Coleman integrals)
2) \( X_0(37)(\mathbb{Q}(i)) \): Daniels and Lozano-Robledo

\[ X_0(37): y^2 = -x^6 - 9x^4 - 11x^3 + 37 \]

over \( \mathbb{Q}(i) \): \( J_0(37)(\mathbb{Q}(i)) \) has rank 2

B-Degra-Müller:

\[ X_0(37)(\mathbb{Q}(i)) = \{ (\pm 2, \pm 1), (\pm i, \pm 4), \infty \} \]

Used QC + Mordell-Weil sieve

\( p = 41, 73, 101 \)

3) \( X_5(13) \): the split Cartan curve of level 13

Bilu-Parent (’11): determined Serre Uniformity in split Cartan case

Bilu-Parent-Rebolledo (’13): determined \( X_5(\ell)(\mathbb{Q}) \) for all \( \ell \neq 13 \)

What about \( \ell = 13 \)?

\( g = 3 \) curve; model was found by Baran (smooth plane quartic)

\( \text{rk} \ NS(J) = 3 \)

B-Degra-Müller-Tulman-Vonk:
\( \text{rk} \ J(\mathbb{Q}) = 3 \)

\( \# X_5(13)(\mathbb{Q}) = 7 \)

Baran \( \Rightarrow \# X_5(13)(\mathbb{Q}) = 7 \)
4) \( X_{s_4}(13) \): \( g=3 \), smooth plane quartic Banwait-Cremona: Jacobian is isog. to \( J \) of \( X_5(13) \)

\[ \# X_{s_4}(13)(\mathbb{Q}) \text{ known} = 4 \]

??

B-Dogra-Müller-Tuitman-Vonk: \( \# X_{s_4}(13)(\mathbb{Q}) = 4 \).

(of interest via Mazur's Program B: last exceptional \( S_4 \) curve; last modular curve of level \( 13^n \))

5) Two other curves from Mazur's Program B (via D. Zureick-Brown)

\[ X_H = X(25)/H \quad \Gamma(25) \subset H \subset GL_2(\mathbb{Z}_5) \]

Each have the following properties:

- 2 known rational points
- usual QC hypotheses satisfied

Fit the global height pairing using the Jacobian and Coleman-Gross p-adic-heights on \( J(\mathbb{Q}) \)

BDMTV ('20): \( \# X_{n}(\mathbb{Q}) = 2 \) \( \} \) used QC+ MWS.
The curves $X_0(N)^+ := X_0(N)/\text{WN}$

are nice curves whose non-cuspidal pts classify unordered pairs $\Sigma E, E_2$ of elliptic curves admitting an $N$-isogeny.

$X_0(N)^+(\mathbb{Q}) = \{ \frac{1}{2} \text{cusps, CM points, exceptional pts} \}$

restrict to $N$ prime.

Galbraith (1996):

$g(\ X_0(N)^+\ ) = 2 \Leftrightarrow N \in \{ 67, 73, 103, 107, 167, 191 \}$

$g(\ X_0(N)^+\ ) = 3 \Leftrightarrow N \in \{ 97, 109, 113, 127, 139, 149, 151, 179, 239 \}$

Y. Hasegawa - K. Hashimoto (1996): $X_0(N)^+$ hyperelliptic $\Leftrightarrow g = 2$.

So $g = 3$ curves are smooth plane quartics.

All satisfy the QC hypotheses:

$J := J_0(N)^+$ has RM, so $\text{rk NS}(J) \geq g$

Can show $\text{rk } J(Q) = g$.

$X_0(N)^+$ has good reduction away from $N$, but $X_0(N)$ does not have potential good reduction at $N$. 
Can show that there's a regular semistable model of $X_0(N)^+$ over $\mathbb{Z}_N$ whose special fiber has a unique irreducible component.

\[ h_N = 0 \]

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Galbraith: What are the exceptional points on $X_0(N)^+$ for all such curves of genus $\leq 5$?

B-Best-Bianchi-Lawrence-Müller-Triantafillou-Vonk:

- $X_0(67)^+ (\mathbb{Q})$: no exceptional points
- $X_0(73)^+ (\mathbb{Q})$, $X_0(103)^+ (\mathbb{Q})$: 1 exceptional pt (up to hyperell. inv.)

BDMTV: The only prime values of $N$ s.t. $X_0(N)^+$ is genus 2 or 3 with exceptional rational pts are $N = 73, 103, 191$.

(So no exceptional points in $g=3$.)

What about $g=4$? Or $g=5$?

ANS 2020: looking at this here.