Quadratic space:

Def. $V = \text{f.d. v-sp over } k$. A quadratic form on $V$ is fn $Q: V \to k$ s.t.

1. $Q(ax) = a^2 \cdot Q(x) \ \forall a \in k, x \in V$
2. the fn $V \times V \to k$
   
   $(x,y) \mapsto Q(x+y) - Q(x) - Q(y)$

   is a bilinear form.

A morphism of quad form $(V, Q) \to (V', Q')$ is a v-sp hom $\Psi: V \to V'$ s.t. $Q' \circ \Psi = Q$.
If $\Psi$ is an isom, often say $\Psi$ is an isometry.

Observe: fn in (2) is symm

- $(x, x) \mapsto Q(2x) - Q(x) - Q(x) = 4Q(x) - 2Q(x) - 2Q(x) = 2Q(x)$.

Assume forever that char $k \neq 2$.

Set $h_Q(x,y) = \frac{1}{2} \left(Q(x+y) - Q(x) - Q(y)\right)$,

$h_Q: V \times V \to k$

Have: \{quad forms $V \to k$ \}$ \leftrightarrow \{\text{symm bilin form } Q \}$

\{ $h_Q \ \ \ V \times V \to k$ \}
Ex: (a) \( V = k \oplus k \)

\[ Q : V \to k , \quad (x_1, y) \mapsto xy \]

\[ h_Q (v_1, v_2) = \frac{1}{2} (Q(v_1+v_2) - Q(v_1) - Q(v_2)) \]

\[ v_1 = (x_1, y_1) \]

\[ v_2 = (x_2, y_2) \]

\[ = \frac{1}{2} ((x_1+x_2)(y_1+y_2) - x_1y_1 - x_2y_2) \]

\[ = \frac{1}{2} (x_1y_2 + x_2y_1) , \]

(b) \( V = \text{Quad field extn of } Q \)

\[ Q : V \to k , \quad x \mapsto \text{Nm}(x) \]

More explicitly: Pick \( d \in k \), squarefree, Set \( V = k(\sqrt{d}) \]

\[ Q(x + y\sqrt{d}) = (x + y\sqrt{d})(x - y\sqrt{d}) = x^2 - dy^2 . \]

What is \( h_Q \)?

Pick a basis \( e_1, \ldots, e_n \) of \( V \).

Define \( A = (a_{ij}) \) by \( a_{ij} = h_Q (e_i, e_j) \)

Symmetric matrix \( h_Q (e_j, e_i) = a_{ij} \)

Change basis by \( X \in \text{GL}_n(k) \), then

\( A \) gets replaced by \( XAX^t \).

Obs: \( \det(A) = \det(A) \cdot \det(X)^2 \)

\( \det(A) \) depends on the choice of basis, but only up to an ess

\( \det (k^2) \).
can define
\[
\text{disc}(Q) := \text{Im} \delta_b \det(A) \text{ in } k^*/(k^*)^2.
\]

**Ex:** Write down an \( A \) for \( \text{Ex}(a), (b) \)

What is the discriminant in these cases?

**Orthogonality.**

\[(V, Q) \xrightarrow{\text{choice } \mathcal{B}} A \text{ symm matrix basis of } V\]

This video: If a choice \( \mathcal{B} \) basis s.t. \( A \) is diagonal.

Fix \( (V, Q) \) quad space.

- \( x, y \in V \) are **orthogonal** if \( h_Q(x, y) = 0 \).
- for any subset \( S \subseteq V \), let
  \[
  S^\perp := \{ v \in V : h_Q(v, s) = 0 \ \forall s \in S \}
  \]
- \( V_1, V_2 \subseteq V \) subspaces. we say \( V_1, V_2 \) are **orthogonal** if \( V_1 \subseteq V_2^\perp \)
- \( V^\perp = \text{orthogonal complement of } V \)
- \( Q \) is **nondegenerate** if \( V^\perp = 0 \).
• $x \in V$ is isotropic if $Q(x) = 0$
• $x \in V$ is anisotropic if $Q(x) \neq 0$
• A quad sp is anisotropic if every nonzero vector is anisotropic.

**Lemma.** If $(V, Q)$ is nondegen, then

\[ V \rightarrow \text{Hom}(V, k) \]
\[ v \mapsto (w \mapsto h_Q(v, w)) \]

is an isomorphism.

**Pf.** If $h_Q(v, w) = 0 \ \forall w \in V$,
then $v \in V^\perp = 0 \Rightarrow v = 0 \Rightarrow$ injectivity.

$\dim V = \dim \text{Hom}(V, k) \Rightarrow$ also get surj. \[\square\]

**Prop.** If $U \subseteq V$ is s.t. $Q|_U$ is nondegenerate,
then $V = U \oplus U^\perp$.

**Pf.** Nondegen $\exists U \ni v \in U \cap U^\perp = 0$.

ETS: $V = U + U^\perp$.
Take $v \in V$ & consider the lin fnl $U \rightarrow k$ $w \mapsto h_Q(v, w)$.
By Lemma, $\exists u \in U$ s.t. $h_Q(v, w) = h_Q(w, u)$ $\forall w \in U$.
$\Rightarrow h_Q(w, v - u) = 0 \ \forall w \in U \Rightarrow v - u \in U^\perp$. \[\square\]
**Thm.** Every quadratic space has an orthogonal basis.

**Pf.** Induct on $\dim V$.

- If $V^\perp = V$, then any basis of $V$ is an orthogonal basis of $V$.
- If $V^\perp \neq V$, then $\exists e_i \in V$ anisotropic.
  - $\Rightarrow ke_i$ is a nondegen. quad. sp.
  - $\Rightarrow$ we can apply Prop $U = ke_i$.
  - $\Rightarrow V = ke_i \oplus ke_i^\perp$
    $\Rightarrow \dim = n - 1$

Let $e_1, \ldots, e_n$ be an orthogonal basis of $V$.
Then the assoc. matr. $A = (a_{ij})$

$$a_{ij} = h_\langle e_i, e_j \rangle = 0 \quad \text{if } i \neq j$$

$\Rightarrow A$ is diag.
Moreover: $\rank A = \# \text{ nonzero els along diag}$

$$= \text{codim of } V^\perp$$

**Ex.** Find an orthogonal basis for Ex(a), (b).
Zero spaces, hyperbolic spaces, and anisotropic space.

In this video:

every quadratic space is a sum of:

- zero space (radical $V^\perp$)
- split space (hyperbolic space)
- nonsplit space (anisotropic part)

Note: orthogonality $\Rightarrow$ can always split off the radical $(V, Q)$ is a fixed nondegen quad space.

Def. A **hyperbolic plane** in a two-dim quadratic space $(H_2, Q)$ s.t. $\exists$ a basis $v_1, v_2 \in H_2$ satisfying:

$Q(v_1) = Q(v_2) = 0$, $h_Q(v_1, v_2) = 1$.

A hyperbolic space of dim $2n \cong (H_2)^\perp \oplus H_2$

Ex: (a) is a hyperbolic plane!

(b) is not anisotropic.
Thm. $V \cong H_{2r} \oplus W$ where $W$ is anisotropic. 
Moreover, such a decomposion is unique up to isom.

With's theorem — see notes & problem set.

Pf of existence. Repeated application of the following key proposition:

**Key prop.** If $V$ contains a $V$ isotropic vector, then $V$ contains a hyperbolic plane.

**Pf.** $x \in V$ nonzero isotropic vector; i.e. $Q(x) = 0$. 
nondeg $\iff \exists y \in V$ s.t. \( hQ(x, y) = 1 \).

Claim: $H := \text{span}\{x, y\}$ is a hyperbolic plane.

**Pf.** Take $v_1 := x$
$v_2 := y + \lambda x$

\[
Q(v_2) = Q(y + \lambda x) = hQ(y + \lambda x)
= hQ(y + \lambda x, y + \lambda x)
= Q(y) + \lambda(hQ(x, y) + hQ(y, x))
= Q(y) + 2\lambda \Rightarrow \lambda = \frac{-Q(y)}{2} \text{ then } Q(v_2) = 0 \]

\[\square\]
Universality:

- A nondegenerate quadratic form $(V, Q)$ is universal if $Q(U) > k^*$.

**Cor.** Any quadratic space with a nonzero isotropic vector is universal.

**Pf.** A hyperbolic plane is universal. \( \Box \)

**Q:** Why do we need nondegeneracy in the car? Is every universal quad space contain a nonzero isotropic vector?

**Hint:** Ex (b).