Automorphic Forms on Unitary Groups

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Plan for series: Introduce aut. forms on unitary gps and some strategies for studying algebraic aspects of L-fcns, esp. in the setting of unitary gps

GOAL FOR TODAY: Motivations + Fundamental Definitions

Motivation from Modular Forms

Example(s):

$\zeta(1-2k)$

$$\zeta(2k) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!} \left(\frac{-B_{2k}}{2k}\right)$$

$k \in \mathbb{Z}_{>0}$

B_{2k} is $2k^{th}$ Bernoulli #

$$\frac{te^t}{e^t-1} = \sum_{n \geq 0}' B_n \frac{t^n}{n!}$$

$\mathbb{Q}$

$$G_{2k}(q) = \zeta(1-2k) + 2\sum_{n\geq 1}' \sigma_{2k-1}(n) q^n$$

$$q = e^{2\pi i z}$$

$$\sigma_{2k-1}(n) = \sum_{d \mid n}' d^{2k-1}$$

- More generally, can pv rationality (up to power of π) of:

  * Dedekind ζ-fcn $\zeta_K(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K}' \frac{1}{N(\mathfrak{a})^s}$

  For K tot'lly real, Klingen and Siegel (based on work of Hecke) pv'd rationality of these vals by realizing as const. terms in F.expn of an E.series ...

..and exploiting properties of space of m. forms.

This approach extends to $L(s,\chi)$ with χ Hecke char of tot'lly real field.

- More generally, could ask ~ algebraicity of certain vals of L-fcns attached to m. forms.

- All these L-fcns agree with corresp. Artin L-fcns

$$L(s, \rho)$$

ρ ← Gal repn

($n=1$ and (part of) $n=2$ cases)

<u>Have conj's ~:</u>

for any n

- Meaning of these L-fcns (Deligne)
- Conn b/w certain ρ and "automorphic repns" (Langlands)

Convenient space to work:

Automorphic forms on unitary gps

Unitary Groups

Fix a CM field K

$$\begin{array}{l} K \\ \mid \text{ quad im} \\ K^+ = \text{totally real} \\ \mid \\ \mathbb{Q} \end{array}$$

and a v.s.

$$V/K$$

with a nondegenerate hermitian pairing $\langle , \rangle$ on V

Rmk: Can extend $\langle , \rangle$ linearly to $V \otimes_{K^+} R$ $\forall$ K^+-alg R

Def: The <u>general unitary group</u>

$GU(V, \langle , \rangle)$ is the alg gp whose R-pts, for each K^+-alg R, are given by

$$G(R) = \left\{ g \in GL_{K \otimes_{K^+} R}(V \otimes_{K^+} R) \;\middle|\; \begin{array}{l} \langle gv, gw \rangle \\ = \nu \langle v, w \rangle \\ \text{some } \nu \in R \end{array} \right\}$$

~~when $\nu = 1$~~
The subgp for which $\nu = 1$ is the <u>unitary group</u> $U(V, \langle , \rangle)$.

Important Case:

If $R = \mathbb{R}$ ~~...~~ and choose an ordered basis for V, can write

$\langle v, w \rangle = v A \left({}^t\bar{w} \right)$ and can choose basis s.t. $A = \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix}$

(a, b) is the <u>signature</u>

For remainder: Assume $K^+ = \mathbb{Q}$ ⑥

Aut. Forms on unitary gps
(and connection with m. forms)

level Γ M. forms of wt k	Aut. Forms on Unitary Gp
① $f: \mathfrak{h} \to \mathbb{C}$ $f(z) = (cz+d)^{-k} f(\gamma z)$ $\forall\ \gamma \in \Gamma \subseteq SL_2(\mathbb{Z})$ holo, ...	have analogous space id'd with G/K_∞ $U(n,m)(\mathbb{R}) / U(n) \times U(m)$ Ex:
② Reformulate as $\phi_f : SL_2(\mathbb{R}) \to \mathbb{C}$ $\phi_f(g) := j(g,i)^{-k} f(g i)$ $\phi_f : \Gamma \backslash G(\mathbb{R}) \to \mathbb{C}$ $SL_2(\mathbb{R}) \circlearrowright \mathfrak{h}$ $SO_2(\mathbb{R})$ fixes i	

② Cont'd

$$\phi_f\left(g\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = e^{ki\theta}\phi_f(g)$$

can extend to

$$\phi_f : \Gamma Z(G)\backslash G(\mathbb{R}) \longrightarrow \mathbb{C}$$

for $G = GL_2, SL_2, GL_2^+$

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$G \rightsquigarrow GU(n,m)$ signature (n,m)

$$\Gamma Z(G)\backslash GU(n,m)$$

$\cup$

stabilizes some pt $\leftarrow$ $K_\infty = U(n)\times U(m)$

$\cap$

$U(n,0)\times U(0,m)$

can define a fcn analogously

$$\Gamma Z(G)\backslash G(\mathbb{R}) \longrightarrow \mathbb{C}$$

③ Adelic interp...

$$GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) \cdot GL_2^+(\mathbb{R}) \times \underbrace{\prod_p K_p}_{=\,K}$$

Cpct open subgps of $GL_2(\mathbb{Q}_p)$ with $\det = \mathbb{Z}_p^\times$ and $= GL_2(\mathbb{Z}_p)$ all but finitely many primes

$$\Gamma := GL_2(\mathbb{Q}) \cap (GL_2^+(\mathbb{R}) \times K)$$

$$\Gamma \backslash GL_2(\mathbb{R}) \longleftrightarrow GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K$$

Get a fcn

$$\varphi_f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \longrightarrow \mathbb{C}$$

$$\varphi_f(\gamma g_\infty (g_p)_p) := \phi_f(g_\infty)$$

arbitrary gp

$$G(\mathbb{A}_f) = \coprod_i G(\mathbb{Q})^+ g_i K$$

can do analogous reformulation in this setting

~~Ex~~

$U(n,n)$ case

An aut. form on $U(n,n)$ is a holo fcn

$$f : \mathfrak{h}_n \to V \circlearrowleft \rho \text{ repn of } GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$$

s.t.

$$f(z) = \rho(Cz+D, \bar{C}\,{}^t z + \bar{D})^{-1} f(\gamma z)$$

$$\forall \quad \gamma \in \Gamma \subseteq U(n,n)(\mathcal{O}_K)$$

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\gamma z = (Az+B)(Cz+D)^{-1}$$

$$\mathfrak{h}_n = \{ z \in M_{n \times n}(\mathbb{C}) \mid i({}^t\bar{z} - z) > 0 \}$$