# Some References

- Piatetski-Shapiro & Rallis
  L-fcns for the Classical Groups.
- Harris
  Shimura varieties for unitary groups and the doubling method
- Garrett
  Pullbacks of Eisenstein series: Applications

  §4.3 manuscript

## Eisenstein series

- Let $P$ be the parabolic subgp of $H$ preserving
$$V^{\Delta} := \{(v,v) \mid v \in V\}$$

- WRT decomp
$$W = V^{\Delta} \oplus V_{\Delta}$$
with $V_{\Delta} := \{(v,-v) \mid v \in V\}$,
have $P = \left\{ \begin{pmatrix} A & * \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix} \middle| A \in GL_n \right\}$

- Given $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}$ Hecke char,

  view as char of $P$ via

$$\begin{array}{ccccc} P & \longrightarrow & GL_n & \xrightarrow{\det} & \mathbb{A}_K^\times \\ \begin{pmatrix} A & * \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix} & \longmapsto & A & \longmapsto & \det(A) \end{array}$$

- Let $s \in \mathbb{C}$ and let
$$f_{s,\chi} \in \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\chi \cdot |\ |^{-s}) = \{ f : H(\mathbb{A}) \to \mathbb{C} \mid f(ph) = \chi(p)|p|^{-s} f(h) \}$$

2 Define <u>Siegel Eisenstein series</u> by

$$E_{f_{s,\chi}}(g) := \sum_{\gamma \in P(\mathbb{Q})\backslash H(\mathbb{Q})}' f_{s,\chi}(\gamma g).$$

$g \in H$

<u>Doubling Integral</u>

- $\pi :=$ cuspidal aut. repn of $G$
- $\tilde{\pi} :=$ contragredient (dual) repn of $\pi$
- $\varphi \in \pi$
- $\tilde{\varphi} \in \tilde{\pi}$
- $$Z(\varphi, \tilde{\varphi}, f_{s,\chi}) := \int_{(G\times G)(\mathbb{Q})\backslash (G\times G)(\mathbb{A})} E_{f_{s,\chi}}(g_1, g_2)\varphi(g_1)\tilde{\varphi}(g_2) \cdot \chi^{-1}(\det g_2)\, dg_1\, dg_2$$
- $Z(\ )$ ~~inherits~~ inherits analytic properties of $E_{f_{s,\chi}}$ (in partic, fcn'l equ, mero cont. as fcn of $s$ to all of $\mathbb{C}$)
- In the case of $G = GU(1)$ (or more generally $GU(n)$ definite unitary gp), can ~~rep~~ re-express as finite sum

<u>Thm:</u> $Z(\varphi, \tilde{\varphi}, f_{s,\chi}) = \int_{G(\mathbb{A})} f_{s,\chi}((g,1)) \langle \pi(g)\varphi, \tilde{\varphi} \rangle \, dg$,

where

$$\langle \varphi, \tilde{\varphi} \rangle = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g) \tilde{\varphi}(g) \, dg$$

$G$-invt pairing, unique up to const. multiple

<u>COR:</u> If $\pi = \otimes_v \pi_v$ and $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ and $\mathrm{Ind}(\chi |\,|^{-s}) = \otimes I_v$ with $\varphi = \otimes_v' \varphi_v$, $\tilde{\varphi} = \otimes_v' \tilde{\varphi}_v$, and $f_{s,\chi} = \otimes_v' f_v$,

then

$$Z(\varphi, \tilde{\varphi}, f_{s,\chi}) = \prod_v Z_v(\varphi_v, \tilde{\varphi}_v, f_v)$$

where

$$Z_v(\varphi_v, \tilde{\varphi}_v, f_v) = \int_{G(\mathbb{Q}_v)} f_v((g,1)) \langle \pi_v(g)\varphi_v, \tilde{\varphi}_v \rangle \, dg$$

<u>Pf of COR:</u> Uniqueness of inv't pairing

$$\Rightarrow \langle \varphi, \tilde{\varphi} \rangle = \prod_v \langle \varphi_v, \tilde{\varphi}_v \rangle .$$

$\blacksquare$

<u>Outline of pf of Thm:</u>

- The thm follows from analysis of orbits of $G \times G$ acting on

$$X := P \backslash H :$$

(identified with $P\gamma$)

- For each $\gamma \in X$, we write $[G \times G]^\gamma$ for the stabilizer of $\gamma$ in $G \times G$.
- For $\gamma \in X$, write $[\gamma]$ for the orbit of $P\gamma$ under action of $G \times G$.

· Write

$$E_{f_{s,\chi}}(h) = \sum_{[\gamma]\, P(\mathbb{Q})\backslash H(\mathbb{Q})/(G\times G)(\mathbb{Q})} \; \sum_{(G\times G)^{\gamma}(\mathbb{Q})\backslash (G\times G)(\mathbb{Q})} f_{s,\chi}(\gamma h)$$

· Insert into doubling integral.

Get

$$\sum{}' \; \sum{}' \left( \int_{(G\times G)(\mathbb{Q})\backslash [G\times G](\mathbb{A})} f_{s,\chi}(\gamma(g,h))\varphi(g)\tilde{\varphi}(h)\chi^{-1}(\det h)\, dg\, dh \right)$$

$$= \sum_{[\gamma]\in P(\mathbb{Q})\backslash H(\mathbb{Q})/(G\times G)(\mathbb{Q})} I(\gamma)$$

with

$$I(\gamma) := \int_{[G\times G]^{\gamma}(\mathbb{Q})\backslash [G\times G](\mathbb{A})} f_{s,\chi}(\gamma(g,h))\varphi(g)\tilde{\varphi}(h)\chi^{-1}(\det g)\, dg\, dh$$

2 cases

- $\gamma = 1$: $I(\gamma) = $ RHS of Thm
- $\gamma \neq 1$: $I(\gamma) = 0$

$$[G\times G]^{\gamma} = \{(g,h) \in G\times G \mid P\gamma(g,h) = P\gamma\}$$
$$= \{(g,h) \in G\times G \mid \gamma(g,h)\gamma^{-1} \in P\}$$

So $[G\times G]^{1} = P \cap G\times G = \{(g,g) \mid g \in G\} =: G^{\Delta}$

- $$f_{s,\chi}(1\cdot(g,h)) = f_{s,\chi}(g,h) = f_{s,\chi}((h,h)\cdot(h^{-1}g,1)) = \chi(\det(h))\cdot f_{s,\chi}(h^{-1}g,1)$$

- So $$I(1) = \int_{G^{\Delta}(\mathbb{Q})\backslash(G\times G)(\mathbb{A})} f_{s,\chi}(h^{-1}g,1)\varphi(g)\tilde{\varphi}(h)\,dg\,dh$$

- $$G\times G \cong G^{\Delta}\times(G\times 1) \cong G\times G$$
$$(g,h) \longleftrightarrow (h,h)\cdot(h^{-1}g,1) \longleftrightarrow (h,h^{-1}g)$$

Let $g_1 = h^{-1}g$

So have

$$I(1) = \int_{G(\mathbb{A})} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} f_{S,x}(g_1, 1)\, \pi(g_1)\varphi(h)\tilde{\varphi}(h)\, dh\, dg_1$$

$$= \int_{G(\mathbb{A})} f_{S,x}(g_1, 1) \langle \pi(g_1)\varphi, \tilde{\varphi} \rangle\, dg_1.$$

$I(\gamma)$ for all the other orbits $[\gamma]$ $(\neq [1])$ all decompose to get product including terms of form

$$\int_{N_i(\mathbb{Q})\backslash N_i(\mathbb{A})} \varphi_i(n \cdot g)\, dn \quad \text{with } i = 1, 2$$

$\varphi_1 = \varphi$, $\varphi_2 = \tilde{\varphi}$ and $N_i$ unipotent radical of a parabolic subgp of $G$ that is nontrivial for at least one $i$.

Can choose $f_{s,\chi}$, $\varphi$, $\tilde{\varphi}$ so that get (nice mults) of Langlands L-fcns $L(\cancel{\pi}_2, s, \pi, \chi)$.

(Relies on reducing computations of local ints to ones computed by Godement & Jacquet)
for $GL_n$