

# Revisiting the doubling method, focusing on $n=1$

GOAL: Let's see what happens  
if we do doubling method  
with  $\mathbb{K}$  imaginary quadratic and  
 $\mathbb{Q}$

$n=1$ :

$V = \overset{\uparrow}{\underset{\sim}{1}}\text{-dim'l v.s / } \mathbb{K}$

$W = V \oplus V$

$$G := U(V, \langle, \rangle) \cong U(1) = \{g \in GL_1 \mid g\bar{g} = 1\}$$
$$\text{in } GU(V, \langle, \rangle) \cong \overset{\sim}{U(1)} \cong GL_1$$

$$H := U(W, \langle, \rangle) \cong \overset{\sim}{U(1, 1)}$$
$$\text{in } GU(W, \langle, \rangle) \cong \overset{\sim}{GU(1, 1)}$$

2

Spoiler: We'll get an expression for  $L(s, \chi)$

with  $\chi: \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$  Hecke char

as finite sum of vals

$$E_\chi(A) \chi(A)$$

for some ~~EG~~ elliptic curves

A with CM by  $(\mathbb{Q}_K)$

and we'll get an algebraicity result.

Rmk: ~~is~~  $GU(1,1) \cong (GL_2 \times \text{Res}_{K/\mathbb{Q}}(G_m)) / G_m$

• Symm space is copies of  $\mathfrak{h}_1 =$  upper half plane

• Aut form is m. form, possibly mild add'l cond on each component

# Doubling Integral

$$Z(s, \chi, \varphi, \tilde{\varphi}) = \int_{(G \times G)(\mathbb{A})} E_{s, \chi}(g, h) \varphi(g) \tilde{\varphi}(h) \chi^{-1}(\det h) dg dh$$

Rmks: ① ~~the~~ Aut form on  $GU(1) \cong GL_1$

is a Hecke char

② If choose  $\varphi = \chi^{-1}$  and so  $\varphi^{-1} = \chi$ ,

get  $Z(s, \chi, \varphi, \tilde{\varphi})$

~~is~~ is actually finite sum

$$\sum_{(G \times G)(\mathbb{A}) / \mathcal{K}} E_{s, \chi}(g, h) \chi^{-1}(g)$$

4

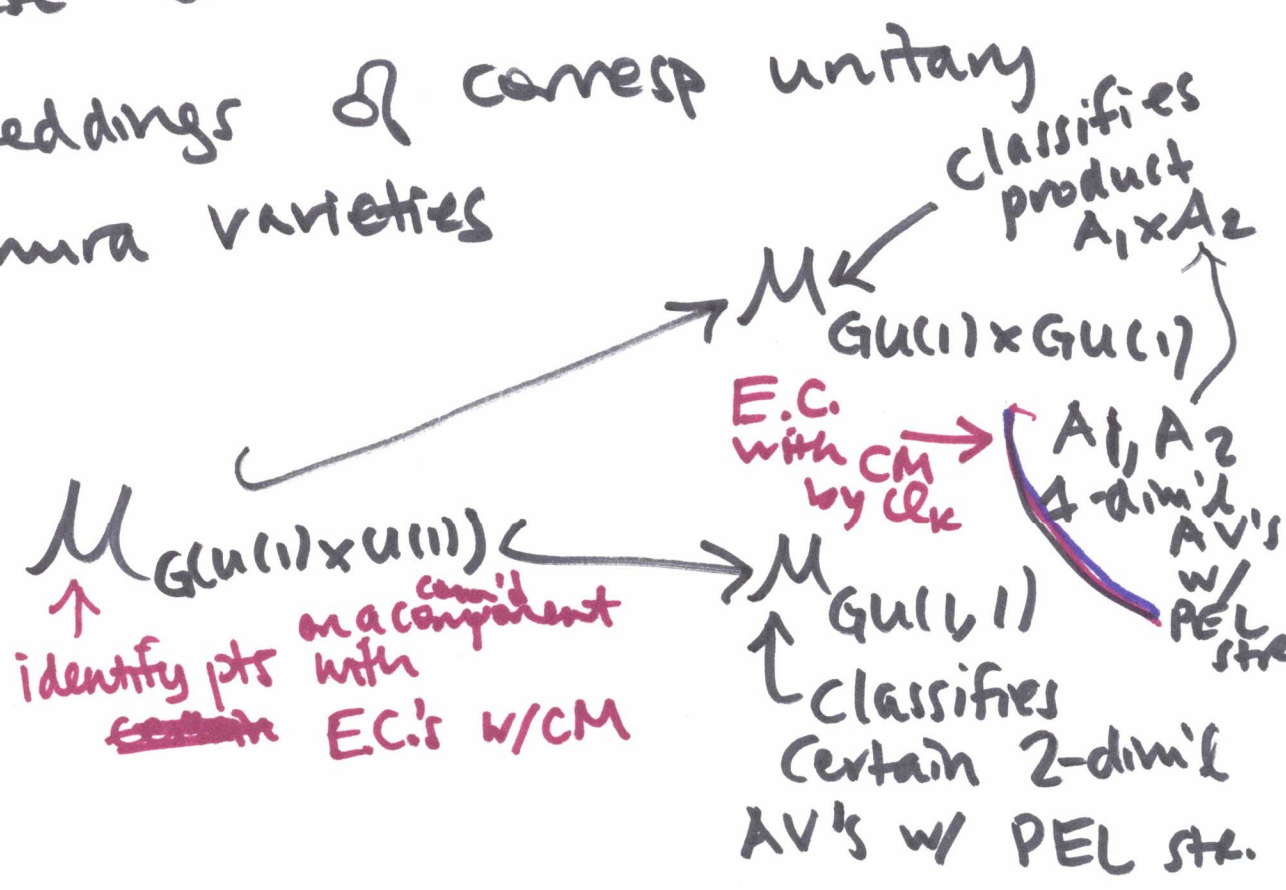
Have embeddings:  $G_4 \times G_1 \cong G_{U(1)} \times G_{U(1)}$

$G(U(1) \times U(1)) \hookrightarrow GU(1,1)$   
 $U(1)$

$U(1) \times U(1) \hookrightarrow U(1,1)$

$G(U(V, \langle, \rangle) \times U(V, -\langle, \rangle)) \hookrightarrow GU(V) \times GU(V) \hookrightarrow GU(W, \langle, \rangle)$   
 $\{(g, h) \in G_U \times G_U \mid \nu(g) = \nu(h)\}$

These embeddings ~~correspond~~ induce embeddings of corresp unitary Shimura varieties





5 Recall: adelic pts of our quotients are the  $\mathbb{C}$ -pts of our unitary sh. varieties, and  $\mathbb{C}$ -pts of  $M_{G(U(1,1))}$  are given by

Rmk: ①  $\prod_{\mathbb{R}} h_i \ni z \longleftrightarrow \mathbb{C} \times \mathbb{C} / \langle (z\bar{a} + \bar{b}, za + b) \rangle$

$\prod_{\mathbb{R}} h_i$

$\left( \begin{array}{c} \updownarrow \\ \mathbb{C} / (z + \bar{z}z) \end{array} \right)$

$a, b \in \mathbb{Q}_K$ -lattice

② Can choose  $f_{s, \chi}$  s.t.

$$Z(s, \chi) := Z(s, \chi, \varphi, \tilde{\varphi}) = (*) L(s, \chi)$$

(i.e. get  $L(s, \chi)$  expressed as a finite sum of vals of

$$E(s, \chi)(\cdot) \cdot \chi(\cdot),$$

with  $E$  is an ~~module~~ ant. form on  $U(1,1)$  (special kind of m. form)

6

This is a variant of

"Damerell's Formula", which  
expresses  $L(s, \chi)$  as finite sum  
of val of  $E(s, \chi) \cdot \chi$

↑  
E. funcs in space of  
Hilbert. m. forms

### Rationality Properties for E. series

- Can obtain an E. series on  $h = h_1$

is of form 
$$\sum_{(g,d) \in \text{appropriate } \mathbb{Q}_k\text{-lattice}} \frac{\chi(d)}{(cz+d)^k |cz+d|^s}$$

Converges for

$$\operatorname{Re}(s) + k > 2$$

- Has rational F. coeffs,  $(2n)$   
when  $s=0$

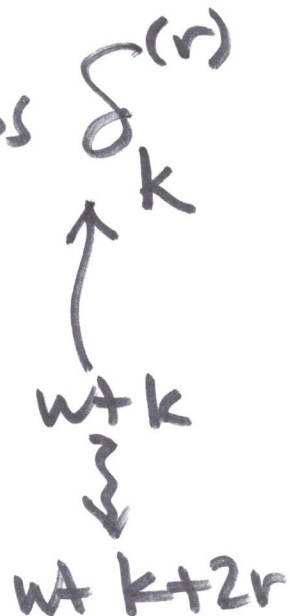
7

Rule:There's a  $q$ -expn principle:"Aut. forms on  $U(n, n)$  are determined by the  $q$ -expns"In partic, if  $q$ -expn coeffs  $\subseteq \mathbb{R}$ ,  
f def'd  $\mathbb{R}$ 

- Kai-wen Lan pr'd <sup>that</sup> for unitary gps, and he showed alg  $q$ -expns and analytic  $q$ -expns agree.

Q: What about  $s \neq 0$ , i.e. when E-series not holo?

A: Use Maass-Shimura diff ops to relate E at  $s \neq 0$  to E at  $s = 0$ .



8. If  $F$  is m. form def'd /  $\mathbb{Q}$ ,  
 then Sh. pr'd

$$\frac{(\delta_k^{(r)} F)(\underline{A})}{\int_{\Omega}^{k+2r} \in \mathbb{Q}}$$

for each CM pt  $\underline{A}$

• These qs have incarnations  
 for  $U(n, m)$  and analogous  
 alg. results

~~forget~~

$$\bullet E(z, -r, \chi) = (*) (-4\pi y)^r \delta_k^{(r)} E(z, 0, \chi)$$

$$\Downarrow$$

Get  $\frac{L(r, \chi)}{\int_{\Omega}^{k+2r} \in \mathbb{Q}}$



9

$$\delta_k f := \frac{1}{2\pi i} \left( \frac{k}{2iy} + \frac{\partial}{\partial z} \right) f$$

$$= \frac{1}{2\pi i} y^{-k} \frac{\partial}{\partial z} (y^k f)$$

$\delta_k^{(r)}$  is compose with itself ~~for~~  $r$  times

Katz's idea:

Re-express this operator geometrically  
over moduli space of E.C.'s  
(or AV.'s)

in terms of Gauss-Manin  
connection and Kodaira-morphism,

$$+ H_{dR}^1 = \underline{\omega} \oplus H^{0,1}$$

↑ preserves alg at CM pts