Revisiting the doubling method, focusing on $n=1$

GOAL: Let's see what happens if we do doubling method with K imaginary quadratic and
$$\begin{array}{c} K \\ | \\ \mathbb{Q} \end{array}$$

$n=1$:

$V = 1$-dim'l v.s $/K$
$$\uparrow$$
$$n$$

$W = V \oplus V$

$$\begin{array}{ccccc} G := U(V, \langle , \rangle_V) & \cong & U(1) = \{ g \in GL_1 \mid g\bar{g} = 1 \} & & \\ \cap & & \cap & & \\ GU(V, \langle , \rangle_V) & \cong & GU(1) & \cong & GL_1 \end{array}$$

$$\begin{array}{ccc} H := U(W, \langle , \rangle) & \cong & U(1,1) \\ \cap & & \cap \\ GU(W, \langle , \rangle) & \cong & GU(1,1) \end{array}$$

<u>Spoiler:</u> We'll get an expression for $L(s,\chi)$

with $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ Hecke char

as finite sum of vals

$$E_\chi(\underline{A})\chi(\underline{A})$$

for some ~~E.C.~~ elliptic curves $\underline{A}$ with CM by $\mathcal{O}_K$)

and we'll get an algebraicity result.

<u>Rmk:</u> $GU(1,1) \cong (GL_2 \times \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m) / \mathbb{G}_m$

- Symm space is copies of $\mathfrak{h}_1$ = upper half plane
- Aut form is m. form, possibly mild add'l cond on each component

Doubling Integral

$$Z(s,\chi,\varphi,\tilde{\varphi}) = \int_{(G\times G)(\mathbb{Q})\backslash(G\times G)(\mathbb{A})} E_{f_{s,\chi}}(g,h)\,\varphi(g)\tilde{\varphi}(h)\,\chi^{-1}(\det h)\,dg\,dh$$

Rmks: ① ~~Gre~~ Aut form on $GU(1) = GL_1$

is a Hecke char

② If choose $\varphi = \chi^{-1}$ and so $\varphi^{-1} = \chi$,

get $Z(s,\chi,\varphi,\hat{\varphi})$

~~[illegible]~~ is actually finite sum

$$\sum_{(G\times G)(\mathbb{Q})\backslash(G\times G)(\mathbb{A})/K} E_{s,\chi}(g,h)\,\chi^{-1}(g)$$

<u>Have embeddings:</u>

$$\begin{array}{ccc} & & GL_1 \times GL_1 \\ & & \| \quad\quad \| \\ & \nearrow & GU(1) \times GU(1) \\ G(U(1)\times U(1)) & \hookrightarrow & GU(1,1) \\ & & \cup \\ U(1)\times U(1) & \hookrightarrow & U(1,1) \end{array}$$

$$\underbrace{G(U(V,\langle,\rangle_V)\times U(V,-\langle,\rangle_V))}_{\{(g,h)\in GU\times GU \mid \nu(g)=\nu(h)\}} \begin{array}{l} \hookrightarrow GU(V)\times GU(V) \\ \hookrightarrow GU(W,\langle,\rangle) \end{array}$$

These embeddings ~~corresp~~ induce ~~to~~ embeddings of corresp unitary Shimura varieties

$$\begin{array}{ccc} & & \mathcal{M}_{GU(1)\times GU(1)} \\ & \nearrow & \\ \mathcal{M}_{G(U(1)\times U(1))} & \hookrightarrow & \mathcal{M}_{GU(1,1)} \end{array}$$

classifies product $A_1 \times A_2$

A_1, A_2 1-dim'l AV's w/ PEL str

E.C. with CM by $\mathcal{O}_K$

identify pts ~~contain~~ on a conn'd component with E.C.'s w/CM

classifies certain 2-dim'l AV's w/ PEL str.

<u>Recall</u>: adelic pts of our quotients are the $\mathbb{C}$-pts of our unitary sh. varieties, and $\mathbb{C}$-pts of $M_{GU(1,1)}$ are given by $\coprod_{\Gamma_x} \mathfrak{h}_1$

<u>Rmk</u>: ① $\mathfrak{h}_1 = \mathfrak{h} \ni z \longleftrightarrow \mathbb{C}\times\mathbb{C} / \langle (z\bar{a}+\bar{b},\ za+b) \rangle$, $a, b \in \mathcal{O}_K$-lattice

$$\updownarrow$$

$$\left(\mathbb{C}/(\mathbb{Z}+\mathbb{Z}z) \right)$$

② Can choose $f_{s,\chi}$ s.t.

$$Z(s,\chi) := Z(s,\chi;\varphi,\tilde{\varphi}) = (*)\, L(s,\chi)$$

(i.e. get $L(s,\chi)$ expressed as a finite sum of vals of
$$E(s,\chi)(\ )\cdot\chi(\ ),$$
with E is an ~~modula~~ aut. form on $U(1,1)$ (special kind of m. form)

This is a variant of "Damerell's Formula", which expresses $L(s,\chi)$ as finite sum of vals of $E(s,\chi)\cdot\chi$

$\uparrow$

E. series in space of Hilbert. M. forms

Rationality Properties for E. series

- Can obtain an E. series on $\mathfrak{h} = \mathfrak{h}_1$ is of form $$\sum_{\substack{(c,d)\in \text{ appropriate} \\ \mathcal{O}_K\text{-lattice}}} \frac{\chi(d)}{(cz+d)^k |cz+d|^s}$$

  Converges for $\mathrm{Re}(s) + k > 2$ ($= 2n$)

- Has rational F. coeffs, when $s=0$

7 Rmk:

There's a q-expn principle:

"Aut. forms on $U(n,n)$ are determined by the q-expns"

In partic, if q-expn coeffs $\subseteq R$, f def'd$/R$

- Kai-Wen Lan pr'd this for unitary gps, and he showed alg q-expns and analytic q-expns agree.

Q: What about $s \neq 0$, i.e. when E. series not holo?

A: Use Maass-Shimura diff ops $\delta_k^{(r)}$ to relate E at $s \neq 0$ to E at $s = 0$.

wt k $\rightsquigarrow$ wt $k+2r$

- If F is m. form def'd $/\overline{\mathbb{Q}}$,

  then Sh. pr'd

$$\frac{(\delta_k^{(r)} F)(\underline{A})}{\Omega^{k+2r}} \in \overline{\mathbb{Q}}$$

  for each CM pt $\underline{A}$

- These qs have incarnations for $U(n,m)$ and analogous

  alg. results

~~So get~~

- $E(z,-r,\chi) = (*)(-4\pi y)^r \delta_k^{(r)} E(z,0,\chi)$

$\rightsquigarrow$ Get $\dfrac{L(r,\chi)}{\Omega^{k+2r}} \in \mathbb{Q}$

$$\delta_k f := \frac{1}{2\pi i}\left(\frac{k}{2iy} + \frac{\partial}{\partial z}\right) f$$
$$= \frac{1}{2\pi i} y^{-k} \frac{\partial}{\partial z}(y^k f)$$

$\delta_k^{(r)}$ is compose δ_k with itself ~~for~~ r times

<u>Katz's idea:</u>

Re-express this operator geometrically over moduli space of E.C.'s (or AV.'s)

in terms of Gauss-Manin connection and Kodaira-morphism, + $H^1_{dR} = \underline{\omega} \oplus H^{0,1}$

preserves alg at CM pts