Revisiting the doubling method, focusing on $n=1$

Goal: Let's see what happens if we do doubling method with $K$ imaginary quadratic and $Q$.

$n=1$:

$V = \frac{1}{2} \text{dim } V \otimes \mathcal{O}_K$

$W = V \oplus V$

$G \coloneqq U(V, <, \geq) \subset U(1) = \{ g \in \text{GL}_1 \mid g \bar{g} = 1 \}$

$\bigcap_{\text{in}} \text{Gu}(V, <, \geq) \cong \text{Gu}(1) \cong \text{GL}_1$

$H \coloneqq U(W, <, >) \cong \Phi U(1, 1)$

$\bigcap \bigcap \text{Gu}(W, <, >) \cong \text{Gu}(1, 1)$
Spoiler: We'll get an expression for \( L(s, \chi) \) with \( \chi: \mathbb{A}_K^\times \to \mathbb{C}^\times \) Hecke char as finite sum of vals

\[ E_x(A) \chi(A) \]

for some \( \mathbb{Q} \)-rational elliptic curves \( A \) with CM by \((\mathbb{Q}_K)\) and we'll get an algebraicity result.

**Rmk:** \( \mathbb{G}U(1,1) \cong \left( \mathbb{G}L_2 \times \text{Res}_{K/F} \mathbb{G}m \right) \)

Symm space is copies of \( \mathbb{H}_1 \) = upper half plane.

Aut form is m-form, possibly mild addl cond on each component.
Doubling Integral

\[ Z(gx, \varphi, \hat{\varphi}) = \int E_{g_x}^*(g, h) \varphi(h) \hat{\varphi}(h) \text{ d}g \text{ d}h \]

\[ (G \times G)(\mathbb{A}) \setminus (G \times G)(\mathbb{A}) \]

Remarks:

1. The automorphic form on \( G U(1) \) is a Hecke character.

2. If choose \( \varphi = x^{-1} \) and so \( \varphi^{-1} = x \),

   get \[ Z(gx, \varphi, \hat{\varphi}) \]

   where is actually finite sum

   \[ \sum E_{g_x}^*(g, h) x^{-1}(g) \]

   \[ (G \times G)(\mathbb{A}) \setminus (G \times G)(\mathbb{A}) \]
Have embeddings: \(GL_1 \times GL_1 \rightarrow GU(1) \times GU(1)\)

\[G(U(1) \times U(1)) \rightarrow GU(1) \times GU(1)\]

\[U(1) \times U(1) \rightarrow U(1,1)\]

\[G(U(V, <, >) \times U(V, -<, >)) \rightarrow GU(V, <, >)\]

\[\{(g, h) \in GU \times GU | \nu(g) = \nu(h)\}\]

These embeddings correspond to embeddings of corresponding unitary Shimura varieties.

Classifies product of \(A_1 \times A_2\) with CM by \(E_k\)

Classifies certain 2-dimensional AV's w/ PEL str.

Identify pts with EC's w/ CM
Recall: adelic pts of our quotients are the C-pts of our unitary sh. varieties, and C-pts of $\mathcal{M}_{\text{GU}(1,1)}$ are given by \[ \bigcup_{\frac{h}{n}} \frac{h}{n} \mathbb{Z} \left\langle (z\bar{a}+b, za+b) \right\rangle \]

Rmk: \[ \frac{h}{n} \mathbb{Z} \longrightarrow \mathbb{C} \times \mathbb{C} \]

\[ \left\langle \left( \frac{a}{(z \bar{a} + b)}, \frac{b}{za+b} \right) \right\rangle \]

\[ a, b \in \mathbb{O}_k \text{-lattice} \]

Can choose \( f, x \) s.t.

\[ \mathcal{E}(s, x) := \mathcal{E}(s, x; \psi, \tilde{\psi}) \]

\[ = \mathcal{E}(s, x) \]

(i.e. get \( \mathcal{E}(s, x) \) expressed as a finite sum of vals of \( \mathcal{E}(s, x) \).

\[ \mathcal{E}(s, x) \cdot \mathcal{X}(x) \]

with \( \mathcal{E} \) is an \textit{automorphic form} on \( U(1,1) \) (special kind of \textit{m. form}).
This is a variant of "Damerell's Formula," which expresses \( L(s, \chi) \) as finite sum of \( \sum \alpha \mathcal{E}(s, \chi) \cdot \alpha \) \( \uparrow \) E-reps in space of Hilbert M. forms

---

Rationality Properties for E-series

- Can obtain an E-series on \( \mathfrak{h} = \mathfrak{h}_1 \)
  - is of form \( \sum \frac{X(d)}{(cz+d)^k |cz+d|^{2s}} \)
    - \((c,d)\)E appropriate \( \mathbb{Q}_k \)-lattice
  - Converges for \( \text{Re}(s) + k > 2n \)
- Has rational F. coeff., when \( s=0 \)
Rank: There's a q-expn principle:

"Aut. forms on U(n, n) are determined by the q-expns"
In part, if q-expn coeffts ≤ R,

f df/dR

- Kai-Wen Lan proved for unitary gps
  and he showed
  alg q-expns and analytic q-expns agree.

Q: What about s=0, i.e. when E-series not holomorphic?

A: Use Maass-Shimura diff. ops \( \varphi_k \)
  to relate \( E \) at \( s=0 \) to \( E \) at \( s=0 \).
8. If $F$ is m. f. m. defined on $\mathbb{R}$, then Sh. pr'd
   $$(S^{(r)}_k)^\infty_0 \frac{e^{zr}}{k+2r} \in \Omega$$
   for each CM pt $A$

   These gs have incarnations
   for $U(n,m)$ and analogous
   alg. results

   \[ E(z, -r, x) = (*) (-4\pi y)^r \sum_k E(z, 0, x) \]

   Get
   $$\frac{L(r, x)}{k+2r} \in \Omega$$
\[ \delta_k f = \frac{1}{2\pi i} \left( \frac{k}{2i'y} + \frac{\partial}{\partial z} \right) f \]

\[ = \frac{1}{2\pi i} \left( y^{-k} \frac{\partial}{\partial z} \right) (y^k f) \]

\( \delta_k \) is composed with itself \( r \) times.

**Katz's idea:**

Re-express this operator geometrically over moduli space of E.C.'s (or A.V.'s) in terms of Gauss-Manin connection and Kodaira-Morphism, $+ H^1 \overset{\text{deg} = 0}{\otimes} H^0$, preserves alg at CM pts.