Automorphic Forms & Theta Correspondence

1. Ramanujan-Petersson Conj.

$f : \mathfrak{h} \longrightarrow \mathbb{C}$

hol. cusp form,

wt k, level 1

Eigenvector for Hecke op. T_p

$$f = \sum_{n>0} a_n(f)\, q^n, \quad q = e^{2\pi i z}$$

$a_1(f) = 1.$

$T_p \cdot f = a_p(f) \cdot f.$

RP Conj (Proved by Deligne)

$$|a_n(f)| \leq 2 \cdot p^{\frac{k-1}{2}}$$

Analog for Maass Forms

$$\left.\begin{array}{l}\text{Hol. Mod Forms}\\ \text{Maass Form}\end{array}\right\} \Rightarrow \begin{array}{l}\text{Aut.}\\ \text{Forms}\end{array}$$

k no. field

v place/prime of k

$\rightsquigarrow$ k_v local field

$\mathbb{A} = \prod_v{}' k_v \overset{\Delta}{\hookleftarrow} k$ ($k \backslash \mathbb{A}$ compact)

G reductive group/k

(eg. SL_N, U_N)

$G(k) \hookrightarrow G(\mathbb{A}) = \prod_v{}' G(k_v)$

wrt $\{K_v\}$ open compact subgps.

$[G] \circlearrowleft G(\mathbb{A})$

$\parallel$

$G(k) \backslash G(\mathbb{A})$

Def: An auto. form on G

is $f : [G] \longrightarrow \mathbb{C}$

satisfying $\begin{cases} \text{regularity} \\ \text{finiteness} \end{cases}$

(smooth, uniform moderate growth.

K-~~finiteness~~, $Z(\mathfrak{g})$-finite)

$$K = \prod_v K_v$$

$$A(G) = \{ \text{auto forms } f \}$$

$$G(\mathbb{A}) \curvearrowright A(G) \qquad (g_0 \cdot f)(g) = f(g g_0)$$

<u>Def</u>: An irred <u>subquot.</u> of $A(G)$ is an auto. rep.

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2 <u>subreps</u>

- Cusp <u>forms</u>

$f \in A(G)$ is cuspidal if $\forall$ parabolic subgp $P = M \cdot N$ the constant term of f along N is 0.

$$f_N(g) = \int_{[N]} f(ng)\, dn$$

$$\mathcal{A}_{cusp}(G) \subseteq \mathcal{A}(G)$$

$$\circlearrowright$$
$$G(\mathbb{A})$$

Rmk: Could take

$$\psi : [N] \longrightarrow \mathbb{C}^\times$$

$\rightsquigarrow$ (N,ψ)-Fourier Coeff of f:

$$f_{N,\psi}(g) = \int_{[N]} \overline{\psi(n)} \cdot f(ng)\, dn$$

(c.f. A. Pollack's Lecture)

Unif. Mod. Growth + Cuspidality $\Longrightarrow$ $f \in \mathcal{A}_{cusp}(G)$ is rapidly $\searrow$ at ∞. $\Longrightarrow$ $\int_{[G]} |f|^2 < \infty$

- $\underbrace{\mathcal{A}_2(G) = \{\text{sq. int. aut. form}\}}$

$$\mathcal{A}_{cusp}(G) \subseteq \mathcal{A}_2(G) \subseteq \mathcal{A}(G)$$

$$\mathcal{A}_{cusp}(G) = \bigoplus_{\pi} m_{cusp}(\pi)\pi \qquad \mathcal{A}_2(G) = \bigoplus_{\pi \in \mathrm{Irr}\, G(\mathbb{A})} m_2(\pi)\pi$$

Q: For which π is

$$m_\bullet(\pi) > 0 \, ?$$

Q: What/How do $\pi \in \mathrm{Irr}\, G(\mathbb{A})$ look like?

Recall: $G(\mathbb{A}) = \prod_v{}' G(k_v)$

$$\rightsquigarrow \quad \pi = \bigotimes_v{}' \pi_v$$

with $\cdot\ \pi_v \in \mathrm{Irr}\, G(k_v)$

$\cdot$ ~~π_v is K~~

$\pi_v^{K_v} \neq 0$ for almost all v.

$\rightsquigarrow$ K_v-unramified/spherical

Unram. Reps

G_v unram. (ie quasi-split over k_v & split by unram ext of k_v)

$\cup$

K_v hyper.-special max compact

$G_v \supset B_v = T_v \cdot N_v$ (Borel)

$$\left\{ \begin{array}{c} K_v\text{-unram} \\ \text{irreps} \\ \text{of } G_v \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{Unram. chars} \\ \text{of } T_v \end{array} \right\} \Big/ \begin{array}{c} \text{Weyl} \\ \text{group} \end{array}$$

$$
\begin{tikzcd}
\left\{\begin{matrix} K_v\text{-unram} \\ \text{irreps} \end{matrix}\right\} \arrow[d, leftrightarrow] & I(\chi) = \operatorname{Ind}_{B_v}^{G_v} \chi & \arrow[l] \begin{matrix} \text{Unique} \\ \text{Unram} \\ \text{Subquot} \\ \text{of} \end{matrix} \\
\left\{\begin{matrix} \text{Unram. chars} \\ \text{of } T_v \end{matrix}\right\} \Big/ W \arrow[d, leftrightarrow, "\text{(Langlands)}"] & \ni \ \chi \arrow[u, mapsto] \arrow[d, mapsto] & \\
\left\{\begin{matrix} \text{semisimple} \\ \text{conj. classes} \\ \text{in } G^\vee(\mathbb{C}) \end{matrix}\right\} & \ni \ s_\chi &
\end{tikzcd}
$$

$$\pi \subseteq \mathcal{A}_{cusp}(G) \longrightarrow \{ s_{\pi_v} \}_{v \notin S} \subseteq G^\vee$$

$$\pi \cong \bigotimes_v{}' \pi_v$$

Aut. L-function,

$$\left\{\begin{array}{l} \pi \subseteq \mathcal{A}_{cusp}(G) \longrightarrow \{s_{\pi_v}\} \curvearrowleft G^\vee \\ R : G^\vee \longrightarrow GL_N(\mathbb{C}) \end{array}\right.$$

$$\rightsquigarrow L^S(s, \pi, R) = \prod_{v \notin S} L(s, \pi_v, R) \qquad (\mathrm{Re}\, s > 1)$$

$$L(s, \pi_v, R) = \frac{1}{\det\left(1 - q_v^{-s} R(s_{\pi_v})\right)}$$

Tempered Reps

Ad-hoc def

$\chi : T_v \to \mathbb{C}^*$

$\downarrow$

π_χ K_v-unram. irrep

Say π_χ is tempered

if χ is unitary

ie $\chi : T_v \to S^1 \subseteq \mathbb{C}^*$

ie $|\chi| = 1$

(Temp. reps are those weakly contained in $L^2(G_v)$)

Reformulation of RP

$\pi \subseteq \mathcal{A}_{cusp}(G)$ (G quasi-split)

$\pi = \bigotimes_v \pi_v$

$\Longrightarrow$ π_v tempered for almost all v.

In Corvallis,

Howe–PS constructed counter-eg for $G = Sp_4$

We'll construct counter eg on $G = U_3$.

Correct: $\pi \subseteq \mathcal{A}_{cusp}(G)$ &

π globally generic.

$\Longrightarrow$ π_v temp. for a.a. v.

Unitary Groups

E/F quad. ext, $\mathrm{Gal}(E/F) = \langle c \rangle$

V v.sp $/E$

$\langle -, - \rangle : V \times V \to E$

ε-Hermitian ($\varepsilon = \pm 1$)

- $\langle a v_1, b v_2 \rangle = a \langle v_1, v_2 \rangle b^c$ $\quad a, b \in E$

- $\langle v_2, v_1 \rangle = \varepsilon \langle v_1, v_2 \rangle^c$

Take $\delta \in E_0^\times = \{ x \in E^\times \mid \mathrm{Tr}(x) = 0 \}$

$\delta \cdot \langle -, - \rangle$ is $-\varepsilon$-Hermitian

$$U(V) = \mathrm{Aut}(V, \langle -,- \rangle)$$

<u>Classification</u> : $\dim V = n$

<u>Invariant</u>

$$\mathrm{disc}(V) = (-1)^{\binom{n}{2}} \det(V)$$

$$\uparrow$$

$$F^\times / NE^\times$$

Henceforth,

V = Herm.

W = skew-Herm