1

Correction

I missed this

ε- Hermitian forms

$$\langle v_2, v_1 \rangle = \varepsilon \langle v_1, v_2 \rangle^c$$

Notation

- V Hermitian, W skew-Herm

- $\mathrm{disc}(V) = (-1)^{\binom{n}{2}} \det(V)$

$(n = \dim V)$ $\quad F^\times / NE^\times$

- $\mathrm{disc}(W) := \mathrm{disc}(\delta^{-n} V)$

$\delta \in E_0^\times$

- <u>p-adic fields</u>

$\left\{\begin{array}{l}\text{disc}(V)\\ \dim(V)\end{array}\right.$ determines V.

$\text{disc}(V) \in F^\times/NE^\times$

$$\begin{array}{c} F^\times/NE^\times \\ \downarrow \wr \; \omega_{E/F} \\ \langle \pm 1 \rangle \end{array}$$

$\longrightarrow V^+, V^-$

- <u>Real</u>

disc not enough

- Need signature (p,q),

$p+q=n$

$$\begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{p} & \\ & \underbrace{-1 \ \ddots \ -1}_{q} \end{pmatrix}$$

$\text{disc}(V_{p,q})$

$= (-1)^q (-1)^{\binom{n}{2}}$

· Number Field: E/k

Local-Global Principle

$$\left\{ \begin{array}{c} \text{Herm. space} \\ \text{over } k \end{array} \right\} \hookrightarrow \prod_v \left\{ \begin{array}{c} \text{Herm. sp.} \\ \text{over } k_v \end{array} \right\}$$

$$V \longmapsto \{ V \otimes_k k_v \}_v$$

Injective!

Image? Given $\{V_v\}_v$,

lies in image (coherent)

$\Updownarrow$

· for a.e. v, $\varepsilon(V_v) = +1$

· $\prod_v \varepsilon(V_v) = +1$.

Eg: p-adic

Rank 1:

$$W_1^+ = \langle \delta \rangle \qquad E_0^\times / N E^\times$$

$$W_1^- = \langle \delta' \rangle \qquad \quad \| \quad$$

$$\qquad \qquad \{ \delta, \delta' \}$$

Rank 2:

$$\mathbb{H} = W_2^+ = E e_1 + E e_2$$

hyperbolic plane

$$\langle e_i, e_i \rangle = 0 \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\langle e_1, e_2 \rangle = 1$$

$W_2^- \to$ des. by quat. div alg.

Rank 2n

$$W_{2n}^{+} = \mathbb{H}^{\oplus n}$$

$$W_{2n}^{-} = W_{2}^{-} \oplus \mathbb{H}^{\oplus (n-1)}$$

Rank 2n+1

$$W_{2n+1}^{+} = \langle \delta \rangle \oplus \mathbb{H}^{\oplus n}$$

$$W_{2n+1}^{-} = \langle \delta' \rangle \oplus \mathbb{H}^{\oplus n}$$

<u>Idea of Howe-PS</u>

$\dim_E W = 3 \rightsquigarrow U(W) \simeq U_3$

$Res_{E/k}(W)$ - 6-dim k-vsp

Symplectic form : $Tr_{E/k}\langle -,- \rangle_W$

$$\begin{array}{ccc} U(W) & \xhookrightarrow{i} & Sp(Res_{E/k}(W)) \\ \| & & \| \\ U_3 & & Sp_6 \end{array}$$

Start with a simple

theta functions $\leftarrow \Omega \subseteq A_2(Sp(-))$

Consider $i^*(\Omega)$

Note:

- Center of $U(W) = E^1$
$= \{x \in E^\times \mid Nx = 1\}$

$i^*(\Omega) = \bigoplus_{\substack{\chi \in E^1 \\ \text{auto char} \\ \text{of } E^1}} \Omega_\chi$ (central char. decom.)

$\cap$
$A(U(W))$

Claim: • Ω_χ is an irred cuspidal rep (except possibly for 1 χ)

• Ω_χ violates RP.

Complication

Theta fns do not live

on Sp_6

but on a cover

Howe - PS produces.

$$\left\{\begin{array}{c}\text{Aut. chars}\\ \text{on } E^1\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Auto reps}\\ \text{of } U_3\end{array}\right\}$$

$$E^1 = U_1$$

$$\chi \longmapsto \Omega_\chi$$

Q: How to produce map,

$$\mathrm{Irr}\, G \longrightarrow \mathrm{Irr}\, H \ ?$$

Simple Idea:

If you have a $G \times H$-rep Ω,

then get a correspondence betw. $\mathrm{Irr}\, G$ & $\mathrm{Irr}\, H$,

i.e. a subset

$$\Sigma_\Omega \subseteq \mathrm{Irr}\, G \times \mathrm{Irr}\, H.$$

(Recall: $\mathrm{Irr}(G \times H) = \{ \pi \otimes \sigma : \pi \in \mathrm{Irr}\, G, \sigma \ldots \}$

$$\Sigma_\Omega = \left\{ (\pi, \sigma) \;\middle|\; \mathrm{Hom}_{G\times H}(\Omega, \pi\otimes\sigma) \neq 0 \right\}$$

$\underline{Q}$: Is this corr. a graph?

$\underline{AH}$:

$$\Omega|_{G\times H} = \bigoplus_\pi \bigoplus_\sigma m(\pi,\sigma)\, \pi\otimes\sigma$$

$$= \bigoplus_\pi \underbrace{\left(\bigoplus_\sigma m(\pi,\sigma)\,\sigma\right)}_{\Theta(\pi)} \otimes \pi$$

$\underline{Q}$: • Is $\Theta(\pi)$ irred. (or 0)?

If so, get $\Theta : \mathrm{Irr}\, G \to \mathrm{Irr}\, H \cup \{0\}$

Point: Need $\dim \Omega$ to be small.

Supp. $G \times H \longrightarrow E$

Take smallest non-triv. rep Ω of E & pull it back to $G \times H$

Would like

$\vartheta : \operatorname{Irr} G \to \operatorname{Irr} H$

to be injective

Theta Corr: an instance of above idea.

F p-adic field E/F quad

V Herm
W skew-Herm $\Big\}$ $V \otimes_E W$ skew-Herm

Get

$$U(V) \times U(W) \longrightarrow Sp(V \otimes_E W)$$

$$G \times H \qquad\qquad\qquad E$$

Ω? To get small enough Ω, need to go to the metaplectic cover.

Metaplectic Group & Weil rep

$$S^1 \longrightarrow Mp(V \otimes W) \twoheadrightarrow Sp(V \otimes W)$$

$$\Big\downarrow \Omega = \omega_\psi$$

$$GL(S)$$

ψ

$\psi : F \longrightarrow \mathbb{C}^*$

$\not\equiv 1$

$\{\omega_\psi\}$ is the smallest inf-dim rep. of Mp. (Weil rep)

Quantum Mechanics

$\Downarrow$

Heisenberg group

$\Downarrow$

Stone - Von-Neumann Thm.

c.f. §§ 2.3
§ 2.4
of Notes

Mp. & ω_ψ

$$\begin{tikzcd} & Mp(V \otimes W) \arrow[d] \\ U(V) \times U(W) \arrow[ur, dashed, "\tilde{i}"] \arrow[r, "i"'] & Sp(V \otimes W) \end{tikzcd}$$

Kudla: $\tilde{i}$ exists & is det

by (χ_V, χ_W) chars of $E^\times$

s.t.

- $\chi_V|_{F^\times} = \omega_{E/F}^{\dim V}$
- $\chi_W|_{F^\times} = \omega_{E/F}^{\dim W}$

Indeed, χ_V gives

$$\tilde{i} : U(W) \longrightarrow Mp$$

χ_W gives

$$\tilde{i} : U(V) \longrightarrow Mp$$

Set

$$\Omega = \Omega_{V,W,\chi_V,\chi_W,\psi}$$

$$= \tilde{\iota}^{*}_{\chi_V,\chi_W}(\omega_\psi)$$

__Properties__: see Lect. Notes

__Def__: For $\pi \in \mathrm{Irr}\, U(V)$,

set $$\Theta(\pi) = \left(\Omega \otimes \pi^\vee\right)_{U(V)}$$

$U(W)$

(Big Θ-lift)

$U(V)$-coinv.

Note:

$$\mathrm{Hom}\left((\Omega \otimes \pi^{\vee})_G, \mathbb{C}\right)$$

$$= \mathrm{Hom}_G(\Omega \otimes \pi^{\vee}, \mathbb{C})$$

$$= \mathrm{Hom}_G(\Omega, \pi)$$

THM (Howe, Kudla)

- $\Theta(\pi)$ has finite length as $U(W)$-rep

  (& so has finitely many irred. quotients)

- For any (π, σ)

$$\dim \mathrm{Hom}_{U(V) \times U(W)}(\Omega, \pi \otimes \sigma) < \infty$$

Def. $\theta(\pi)$ = cosocle of $\Theta(\pi)$

small theta lift = max. semisimple quotient

Thm (Howe Duality)

- $\Theta(\pi)$ is irred. if $\Theta(\pi) \neq 0$
- $\Theta(\pi) \cong \Theta(\pi') \Rightarrow \pi \cong \pi'$

Get

$$\theta : \mathrm{Irr}\, U(V) \longrightarrow \mathrm{Irr}\, U(W) \cup \{0\}$$

injective on

$$\operatorname{supp} \Theta = \mathrm{Irr}\, U(V)_{\neq 0}$$

$\boxed{\text{Q: Is } \Theta(\pi) \text{ zero or not?}}$