Correction : HOWE-PS

$$\left\{ \begin{matrix} \text{Auto char} \\ \text{of } U_1 \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Auto Rep} \\ \text{of } U_3 \end{matrix} \right\}$$

$$\chi \longmapsto \Omega_\chi$$

" Ω_χ is cuspidal except possibly for at most <u>one</u> χ "

$\boxed{\text{Nonesense}}$

RECALL: Weil rep of $U(V) \times U(W)$

$$\Omega = \Omega_{V, W, \chi_V, \chi_W, \psi}$$

For $\pi \in \mathrm{Irr}\, U(V)$,

$$\Theta(\pi) = \left(\Omega \otimes \pi^\vee\right)_{U(V)} \circlearrowright U(W)$$

HOWE DUALITY : If $\Theta(\pi) \neq 0$, has unique irred quot $\theta(\pi)$

(same for $\sigma \in \mathrm{Irr}\, U(W)$)

$\rightsquigarrow \theta : \mathrm{Irr}\, U(V) \longrightarrow \mathrm{Irr}\, U(W) \sqcup \{0\}$
injective outside zero locus

Q : When is $\Theta(\pi) \neq 0$?

dim W = odd , dim $W_r^{\varepsilon} = 2r+1$

$$
\begin{tikzcd}
U(W_r^-) & & U(W_r^+) \\
\vdots & U(V) \arrow[ul, "\theta_r^-"'] \arrow[dr, "\theta_1^+"] & \vdots \\
& \pi & \\
U(W_1^-) & & U(W_1^+) \\
U(W_0^-) & & U(W_0^+)
\end{tikzcd}
$$

(Rallis' tower)

Note : $W_{r+1}^+ = W_r^+ \oplus \mathbb{H}$

Q : Which of these lifts

$\theta_r^{\varepsilon}(\pi)$ are nonzero?

THM:

(1) For $\pi \in \mathrm{Irr}\, U(V)$, & fixed $\varepsilon = \pm$

$\exists$ a smallest $r_0 = r_0^{\varepsilon}(\pi) \leq \dim V$

st $\Theta^{\varepsilon}_{r_0^{\varepsilon}(\pi)}(\pi) \neq 0$.

(first occurrence of π in the ε-tower.)

(2) $\forall\ r > r_0$, $\Theta^{\varepsilon}_{r}(\pi) \neq 0$

(3) π supercuspidal rep

$\Rightarrow$ (Kudla) $\Theta^{\varepsilon}_{r}(\pi)$ is irred.

& is s.c. at the first occurrence

(but not after)

Q: Non-vanishing is reduced to determining

$$r_0^+(\pi) \quad \& \quad r_0^-(\pi)$$

Rmk:

- If $r \geq \overset{\dim V}{\cancel{r_0}}$, $\Theta_r^{\varepsilon}(\pi) \neq 0$

stable range

Thm (Conservation Relation)

$$\dim W^+_{r_0^+(\pi)} + \dim W^-_{r_0^-(\pi)} = 2 \dim V + 2 \quad \forall \pi$$

(B.Y Sun & C.B. Zhu
Kudla-Rallis)

Cor (Dichotomy)

If $\dim W^+ + \dim W^-$

$= 2 \dim V$

then for any $\pi \in \mathrm{Irr}\, U(V)$,

exactly one of

$\Theta_{W^+}(\pi)$ or $\Theta_{W^-}(\pi)$

is nonzero

Eg: $U_1 \times U_1 = U(V) \times U(W_0)$

$U(V) = E^1, \quad \chi \in \mathrm{Irr}\, E^1$

$$\dim W^{+}_{r^{+}(\chi)} + \dim W^{-}_{r^{-}(\chi)} = 4$$

$\searrow \quad \swarrow$

$\{1, 3\}$

Exactly one of

$\theta_0^{+}(\chi)$ or $\theta_0^{-}(\chi)$

is non-zero

& $\forall\, r > 0, \ \theta_r^{\varepsilon}(\chi) \neq 0.$

Q: Which of $\theta_0^{\varepsilon}(\chi)$

is non-zero?

THM (MOEN, ROGAWSKI, Harris-Kudla-Sweet)

$$\Theta_{V, W_0, \psi}(\chi) \neq 0 \qquad \chi \in \mathrm{Irr}\, E^1 = \mathrm{Irr}\, U(V)$$

$$\Updownarrow$$

$$\varepsilon(V) \cdot \varepsilon_\delta(W_0)$$

$\in \pm 1$

$$= \varepsilon_E\left(\tfrac{1}{2}, \chi_E \cdot \chi_W^{-1}, \psi(\mathrm{Tr}_{E/F}(\delta -))\right)$$

Tate's local ε-factor

$$\begin{tikzcd} E^\times/F^\times \arrow[r, "\sim"] \arrow[rr, bend left, "\chi_E"] & E^1 \arrow[r, "\chi"] & \mathbb{C}^\times \end{tikzcd}$$

$$x \longmapsto x/x^c$$

$$\chi_E(x) = \chi\left(\frac{x}{x^c}\right)$$

$\delta \in E_0^\times$

Apply to Howe-PS : $U_1 \times U_3$

$V = \langle 1 \rangle = V_0^+$, $\chi \in \mathrm{Irr}\, E^1 = \mathrm{Irr}\, U(V)$

$\dim W_1^\varepsilon = 3$

$$\Omega^\varepsilon = \bigoplus_{\chi \in \mathrm{Irr}\, E^1} \chi \otimes \Theta^\varepsilon(\chi)$$

$U(W)$

- $\Theta^\varepsilon(\chi) \neq 0 \quad \forall \chi$. (stable range)
- $\Theta^\varepsilon(\chi)$ is irred (Howe duality + s.c.)
- If $\varepsilon = \varepsilon_E(\frac{1}{2}, \text{——})$, as in Thm [HKS]

$\Theta^\varepsilon(\chi)$ non-sc $\to$ in fact,

$\Theta^{-\varepsilon}(\chi)$ s.c

$$\Theta^\varepsilon(\chi) \hookrightarrow \mathrm{Ind}_B^{U(W)} \chi_V |\cdot|_E^{-\frac{1}{2}} \otimes \chi$$

non-temp.

$$B = \left\{ \begin{pmatrix} a & & * \\ & b & \\ 0 & & (a^c)^{-1} \end{pmatrix} \right.$$

$a \in E^\times$, $b \in E^1$

Global <u>setting</u> : k number field

$$\theta = \bigotimes_v \theta_v : \mathrm{Irr}\, U(V)_{/\mathbb{A}} \longrightarrow \mathrm{Irr}\, U(W)_{/\mathbb{A}}$$

(abstract lifting)

<u>Want</u> :

$$\theta : \left\{ \begin{matrix} \text{Auto rep} \\ \text{of } U(V) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Aut rep} \\ \text{of } U(W) \end{matrix} \right\}$$

How to transfer functions

from space X to Y

<u>Simple</u> : If $K \in C(X \times Y)$

then get

$$T_K : C(X) \longrightarrow C(Y)$$

$$T_K(f)(y) = \int_X K(x, y) f(x)\, dx$$

In our <u>case</u>:

$$\begin{tikzcd} & Mp(V \otimes W)_{/\mathbb{A}}^{\omega_\psi} \\ U(V)_{/\mathbb{A}} \times U(W)_{/\mathbb{A}} \arrow[ur, dashed, "\tilde{\iota}"] \arrow[r, "\iota"] & Sp(V \otimes W)_{/\mathbb{A}} \end{tikzcd}$$

$\Omega = \tilde{\iota}^* \omega_\psi$

Have:

$$\begin{tikzcd} \theta : \omega_\psi \arrow[r] & A_2(Mp(-)) \arrow[d, "\tilde{\iota}^*"] \\ \phi \arrow[u, phantom, "\in"] & \end{tikzcd}$$

$\pi \in A_{cusp}(U(V))$ $\quad \theta(\phi) \in C([U(V) \times U(W)])$

Get:
$$\omega_\psi \otimes \pi \longrightarrow A(U(W))$$
$$\phi \otimes f \longmapsto \Theta(\phi, f)$$

$$\Theta(\phi, f)(g) = \int_{[U(V)]} \theta(\phi)(g, h)\, \overline{f(h)}\, dh$$

Set (Global Θ-lift of π)

$$\Theta(\pi) = \left\langle \theta(\varphi, f) : \begin{array}{l} \varphi \in \omega_\psi \\ f \in \pi \end{array} \right\rangle$$

$$\cap\!\!|$$

$$A(U(W))$$

<u>Q</u>: · Is $\Theta(\pi)$ nonzero?

· Is $\Theta(\pi) \subseteq A_2(U(W))$

$A_{cusp}(U(W))$

· Relation with local?

<u>Prop</u>: If $\Theta(\pi) \subsetneq A_2(U(W))$.

then $\Theta(\pi)$ is either 0

or $\Theta(\pi) \simeq \bigotimes_v \theta(\pi_v)$

THM : $\pi \subseteq \mathcal{A}_{cusp}(U(V))$

(i) $\exists$ smallest $r_0 = r_0^{\varepsilon}(\pi)$ st $\Theta_{r_0}^{\varepsilon}(\pi) \neq 0$.

$r_0^{\varepsilon}(\pi) \leq \dim V$

in which case

$$\Theta_{r_0}^{\varepsilon}(\pi) \subseteq \mathcal{A}_{cusp}(U(W))$$

(ii) $\forall\ r > r_0$, $\Theta_r^{\varepsilon}(\pi) \neq 0$ & non-cuspidal (ie $\not\subseteq \mathcal{A}_{cusp}(U(W))$)

stable range

(iii) $\forall\ r \geq \dim V$

$$0 \neq \Theta_r^{\varepsilon}(\pi) \subseteq \mathcal{A}_2(U(W)).$$

by (i) & (ii)