Addendum

Deg Eis series:

on G_2: $E_\ell(g,f,s) = \sum_{\gamma \in P(\mathbb{Q})\backslash G_2(\mathbb{Q})} f_\ell(\gamma g, s)$

ℓ even

<u>$s=\ell+1$:</u> $E_\ell(g, f, s=\ell+1)$: a wt ℓ MF

A fair amount of work: $D_\ell f_\ell(g, s=\ell+1) = 0$.

Project gp: similar statement for $U(2,n)$

F.C.'s of $E_\ell(g, f, s=\ell+1)$:

nonzero when $\ell = 4$ by Gan's Siegel-Weil Thm

$$\begin{tikzcd}[column sep=small]
& \Theta \quad E_{8,4} \arrow[dl, dash] \arrow[dr, dash] & \\
G_2 & & F_4^{an} \\
E_4(g) & & 1 \arrow[ll]
\end{tikzcd}$$

Lecture 4 Beyond G_2

Upshot: • $\exists$ gps $\underbrace{G_2, F_4}_{\text{split}}, \underbrace{E_{n,4}}_{\text{rk } 4 / \mathbb{R}}$ $n = 6, 7, 8$

- $\exists$ MFs on these gps, w/ F.E & F.C.'s similar to G_2 story
- One can a little bit about F.C.s of MFs on these bigger gps

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Exceptional Algebra

- C = composition alg / k = char 0
- $\exists$ mult $C \otimes C \to C$ not necc comm, assoc.
- $\exists$ $n_C : C \to k$ non-deg quad form

  w/ $n_C(x \cdot y) = n_C(x)\, n_C(y)$

Examples

$C = k : \quad n_C : k \to k \quad x \longmapsto x^2$

$C = E/k :$ E quad étale extn

$$n_C = n_{E/k}$$

$C = B/k :$ B quat alg, $n_C = n_{B,\mathrm{red}}$

$C = \mathbb{O} :$ $\mathbb{O}$ oct. alg, $\mathbb{O} = B \oplus B$

$\rightsquigarrow \quad {}^*: C \to C$ involution s.t.

$x + x^* \in k \cdot 1, \quad x + x^* = \mathrm{tr}_C(x)\, 1$

$x \cdot x^* \in k \cdot 1, \quad x \cdot x^* = n_C(x)\, 1$

<u>Define</u> $J_C = H_3(C) =$ Hermitian 3×3 matrices w/ coeffs in C

$$= \left\{ \overbrace{\begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix}}^{X} : c_i \in k,\ x_i \in C \right\}$$

$\dim_k = 3 + 3\cdot C$

<u>Ex:</u> $C = k$ $\quad H_3(k) =$ Sym 3×3 matrices

$\exists$ $\det : J_C \to k$ as

$$\det(X) = c_1c_2c_3 - c_1 n_C(x_1) - c_2 n_C(x_2) - c_3 n_C(x_3) + \mathrm{tr}_C\big(x_1(x_2x_3)\big)$$

<u>Rmk 1</u> If $C = k$ this is the usual det on 3×3 sym matrices

<u>Rmk 2</u> $M^1_J = \{ g \in GL(J_C) : \det(gX) = \det(X)\ \forall X \in J_C \}$ has pos dim^{n}.

<u>Idea:</u> $\exists$ gp G_{J_C} s.t. if

$C = \mathbb{Q}$, $\quad G_{J_C} = F_4$

$C = \mathbb{K}$, $\quad G_{J_C} = E_{6,4}$
quad imag

$C = \mathbb{B}$, $\quad G_{J_C} = E_{7,4}$
quad alg
ram @ ∞

$C = \mathbb{O}$, $\quad G_{J_C} = E_{8,4}$
oct alg
has pos def n_C

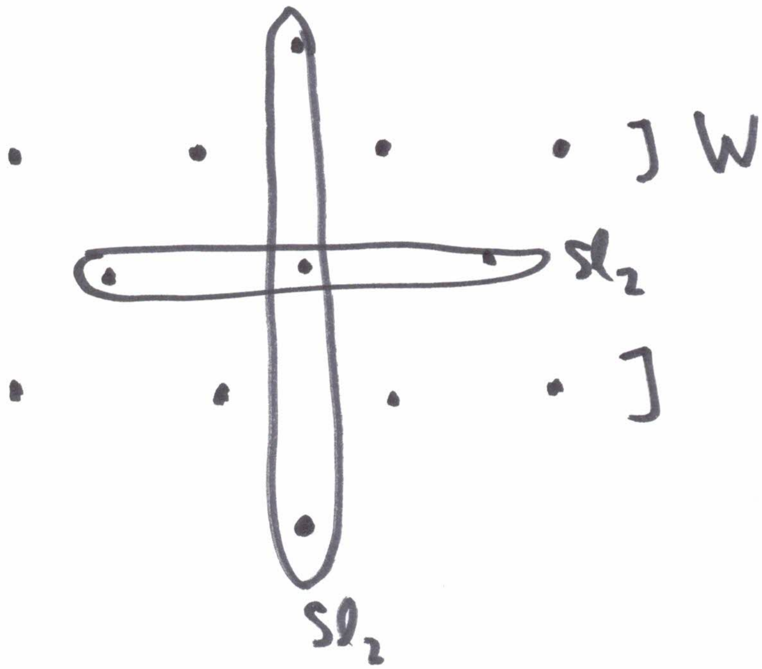
- all of these gp have QMFs
- QMFs have F.E.'s & F.C.'s

<u>Rmk:</u>

$$\begin{tikzcd} J_C \arrow[r, squiggly, "\text{``deg''}"] \arrow[d, "\det"] & k \arrow[d, "x \mapsto x^3"] \\ k \arrow[r, squiggly] & k \end{tikzcd}$$

$$G_{J_C = k} = G_2$$

Recall:



$$\mathfrak{g}_2 = \underbrace{\mathfrak{sl}_{2,l} + \mathfrak{sl}_{2,s}}_{\deg 0} + \underbrace{V_2 \otimes W}_{\deg 1}$$ (a $\mathbb{Z}/2$-grading)

I'll mimic this $\mathbb{Z}/2$ grading to construct a Lie alg $\mathfrak{g}_{J_e}$

Freudenthal construction

- Let $J = J_C$ / $\mathbb{Q} = k$
- $W_J = \mathbb{Q} \oplus J \oplus J^\vee \oplus \mathbb{Q}$
- $\exists$ symp form

$$\langle (a,b,c,d), (a',b',c',d') \rangle$$
$$= ad' - (b,c') + (c,b') - da'$$

- $\exists\ q : W_J \to \mathbb{Q}$ deg 4 poly map

Define $H'_J = \{ g \in GL(W_J) : \langle gw, gw' \rangle = \langle w, w' \rangle\ \forall w, w' \in W_J,\ q(gw) = q(w) \}$

C	H'_J	$\mathfrak{g}_J$
$\mathbb{Q}$	C_3	F_4
K	A_5	E_6
B	D_6	E_7
$\mathbb{O}$	E_7	E_8

Define

$$\mathfrak{g}_J = \underbrace{\mathfrak{sl}_2 + \overset{\overset{\mathrm{Lie}(H_J')}{\parallel}}{\mathfrak{h}_J^0}}_{\deg 0} + \underbrace{V_2 \otimes W_J}_{\deg 1}$$

$\mathfrak{g}_J$ has structure of Lie alg

$G_J := \mathrm{Aut}^0(\mathfrak{g}_J)$

Def^n Suppose $n_C : C \otimes \mathbb{R} \to \mathbb{R}$ is pos def.
Then we call G_J a quat. exc. gp

Fact: $K_J \leq G_J(\mathbb{R})$ max'l cmpt subgp. Then

$$K_J = (SU(2) \times L_J')/\mu_2, \quad L_J' \text{ compact form of sp } H_J'$$

$\Theta : \mathfrak{g}_J \to \mathfrak{g}_J$ be cartan involution

$$\mathfrak{g}_J^{\Theta=1} \otimes \mathbb{C} = \mathfrak{k}_0 \otimes \mathbb{C} \simeq \mathfrak{sl}_2 + \mathfrak{h}_J^0 \quad / \mathbb{C}$$

$$\mathfrak{g}_J^{\Theta=-1} \otimes \mathbb{C} = \mathfrak{p}_0 \otimes \mathbb{C} \simeq V_2 \otimes W_J \quad / \mathbb{C}$$

$K_J \curvearrowright \mathbb{V}_\ell = \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \boxtimes \mathbb{1}$

<u>Defn</u> A mod form on G_J of wt ℓ is an aut form

$$\varphi : G_J(\mathbb{Q}) \backslash G_J(\mathbb{A}) \to \mathbb{V}_\ell \quad \text{s.t.}$$

① $\varphi(gk) = k^{-1} \cdot \varphi(g) \quad \forall\, k \in K_J$

② $D_\ell \varphi \equiv 0$ ← defined exactly as before, replacing $\mathrm{Sym}^3(V_2) = W$ in the G_2 case w/ W_J

$G_J \supseteq P = MN$ Heisenberg parabolic

w/ $M \simeq H_J$ (sim. version of H_J^1)

$N \supseteq Z$ two step w/

Z: 1-dim'l

$N/Z \cong W_J$ is abelian

<u>Thm:</u> Mod forms on G_J of wt ℓ have

1. F.E / F.C.s along N/Z

$$\varphi_Z(g) = \varphi_N(g) + \sum_{\substack{w \in W_J(\mathbb{Q}) \\ w \geq 0}} a_\varphi(w) W_w(g)$$

- $a_\varphi(w) \in \mathbb{C}$ are the F.C.'s of φ
- W_w are completely explicit

2. • If $G_2 \leq F_4 \leq E_{6,4} \leq E_{7,4} \leq E_{8,4}$ are embedded appropriately, φ mod form of wt ℓ one gp, $\iota^*(\varphi)$ the pull back to smaller gp

<u>THEN:</u> • $i^{\circ}(\varphi)$ is a MF of wt l

• F.C.s of $i^{\circ}(\varphi)$ are <u>fte Σ's</u> of the F.C.s of φ

<u>Thm:</u>

(Can, P.) ① $\exists$ nonzero wt 4 MF on $E_{S,4}$, all of whose F.C.s $\in \mathbb{Q}$: $\mathcal{E}_{min}$

(Savin, P.) ② $\exists$ nonzero wt 8 MF on $E_{S,4}$ all of whose F.C.'s $\in \mathbb{Q}$: $\mathcal{E}_{ntm}$

<u>Pf:</u> (Sketch of 2)

a) Construct $\mathcal{E}_{ntm}$ using Eis series

b) <u>Savin:</u> most of F.C.s of $\mathcal{E}_{ntm}$ are 0

c) "Explicit comp": the other F.C.'s $\in \mathbb{Q}$

<u>Def$^{\underline{n}}$</u> Say a MF: φ on G_J is **distinguished** if

(1) $\exists\ w_0 \in W_J(\mathbb{Q})$ s.t. $q(w_0) \neq 0$ and $a_\varphi(w_0) \neq 0$

(2) if $w \in W_J(\mathbb{Q})$, $a_\varphi(w) \neq 0$ then $q(w) \equiv q(w_0) \mod (\mathbb{Q}^\times)^2$

<u>Thm:</u> Suppose $K/\mathbb{Q}$ quad imag. $\exists$ a distinguished wt 4 MF Θ_K on $G_{J_K} = E_{6,4}$.

<u>Pf:</u> · $\Theta_K := i^*(\Theta_{min})$ pull back to $E_{6,4}$

· AIT $\Rightarrow$ Θ_K is distinguished.