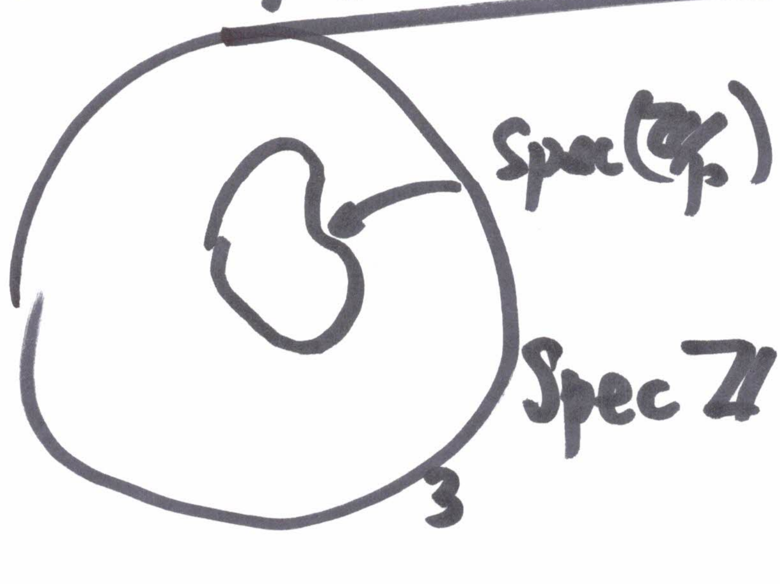
Number fields $\xleftrightarrow[\text{topology}]{\text{arithmetic}}$ 3 manifolds

automorphic forms $\longleftrightarrow$?

Mazur (63/64) Artin Tate Mumford

"Spec $\mathbb{Z}/p$ is like a knot in Spec $\mathbb{Z}$, which is like a simply connected 3-manifold."



Weil (1949)

... should be an "algebraic" cohomology for varieties X / K (alg. closed)

$$X \rightsquigarrow H^*(X)$$

such that, for $K = \mathbb{C}$, recovers singular cohomology of top. space $X(\mathbb{C})$.

e.g. $X \quad \{x^3 + y^3 + z^3 = 0\} / \mathbb{C}$

$x \leftrightarrow y$ act on $H^*(X)$.

$x \to \bar{x}, y \to \bar{y}, z \to \bar{z}$ act on $H^*(X)$

$x \to \sigma(x), y \to \sigma(y), z \to \sigma(z)$ should act on $H^*(X)$?!

Weil's proposal realized by Artin & Grothendieck:

(for finite coefficients)

étale cohomology $H^*(X)$

Tate (1962) ⇝ showed

Poitou (1961)

étale cohomology of Spec $\mathbb{Z}$ (or other number rings) has duality $H^i \longleftrightarrow H^{3-i}$.

H^*(manifold) **Example**

linear algebra

$0 \to \text{Vertices} \to \text{Edges} \to \text{Faces}$

$H^*(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z})$.

$H^1 \simeq \dfrac{\text{units in } \mathbb{Z}(\frac{1}{2})}{\text{squares}} = \{\pm 1, \pm 2\}$

$(\simeq (\mathbb{Z}/2)^2)$.

$H^2 \simeq \left\{ \begin{array}{l} \text{"quaternion"} \\ \text{algebras over } \mathbb{Z}[\frac{1}{2}] \end{array} \right\} \simeq (\mathbb{Z}/2)$

$M_2(\mathbb{Z}[\tfrac{1}{2}])$ ↗ $\qquad$ ↖ $\mathbb{Z}[\tfrac{1}{2}](i, j : i^2 = -1, j^2 = -1, ij = -ji)$

Compare duality for number ring & 3-manifold.

Number ring $\mathcal{O}$ = { S-integers in a number field, or functions on a smooth curve / $\mathbb{F}_q$

For simplicity: $\mathbb{Z}[\frac{1}{p}]$.

$$H^i(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{p}], M)$$

↑ p-torsion abelian group e.g. $\mathbb{Z}/p^n\mathbb{Z}$ (can have Galois action unram. outside p)

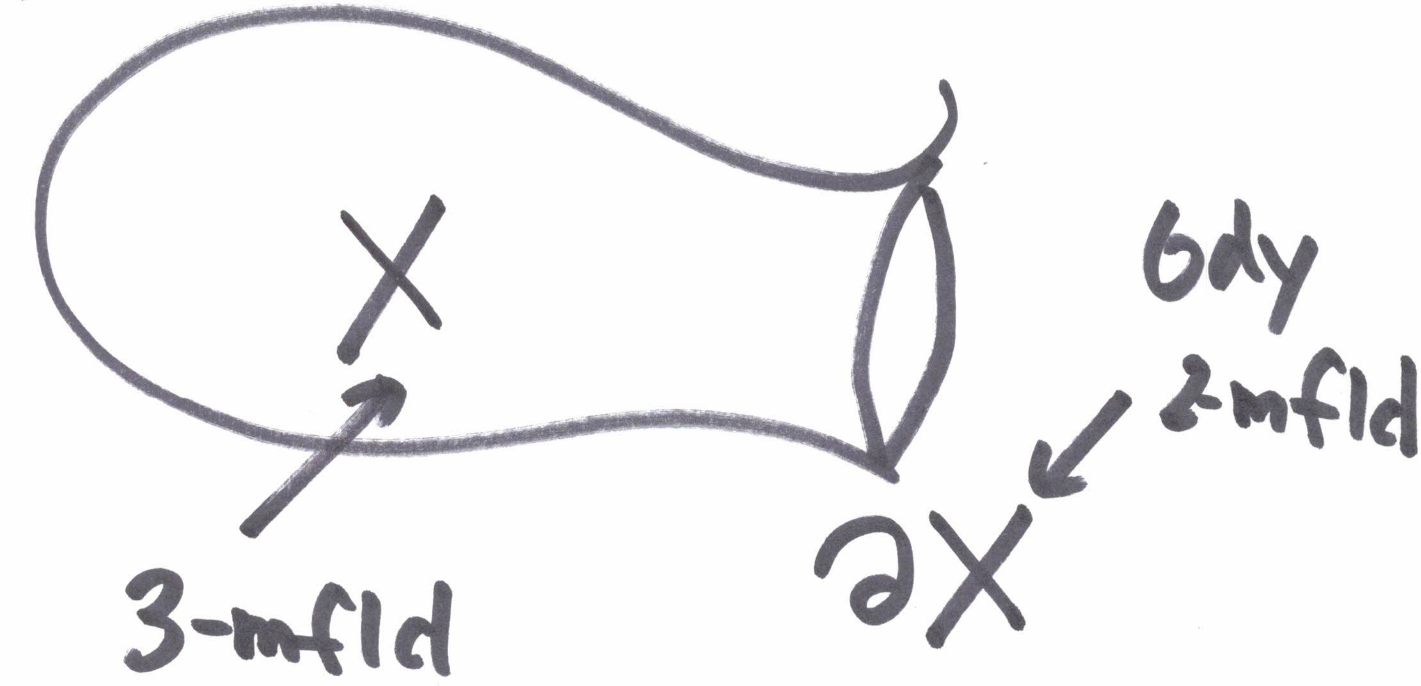
Tate duality

{degree i-1}

$$(\quad) \to H^i(\mathbb{Z}[\tfrac{1}{p}], M) \to H^i(\mathbb{Q}_p, M)$$

$$\uparrow \qquad\qquad\qquad\qquad \downarrow$$

$$H^{3-i}(\mathbb{Z}[\tfrac{1}{p}], M^*)^* \qquad\qquad \{\text{degree } i+1\}$$



$$\begin{array}{c} (i-1) \\ \downarrow \end{array}$$

$$H^{3-i}(\mathbb{Z}[\tfrac{1}{p}], M^*)^* \longrightarrow H^i(\mathbb{Z}[\tfrac{1}{p}], M) \longrightarrow H^i(\mathbb{Q}_p, M)$$

$$\begin{array}{c} \downarrow \\ (i+1) \end{array}$$

$M^* = \{\text{hom. } M \to \text{circle}\}$

$= \{\text{hom. } M \to \text{p-power roots of } 1\}$

3-manifold X with bounday ∂X

$$H^i(X, \partial X; M) \longrightarrow H^i(X, M) \longrightarrow H^i(\partial X, M)$$

$$\wr\!\wr$$

$$H^{3-i}(X; M^*)^*$$

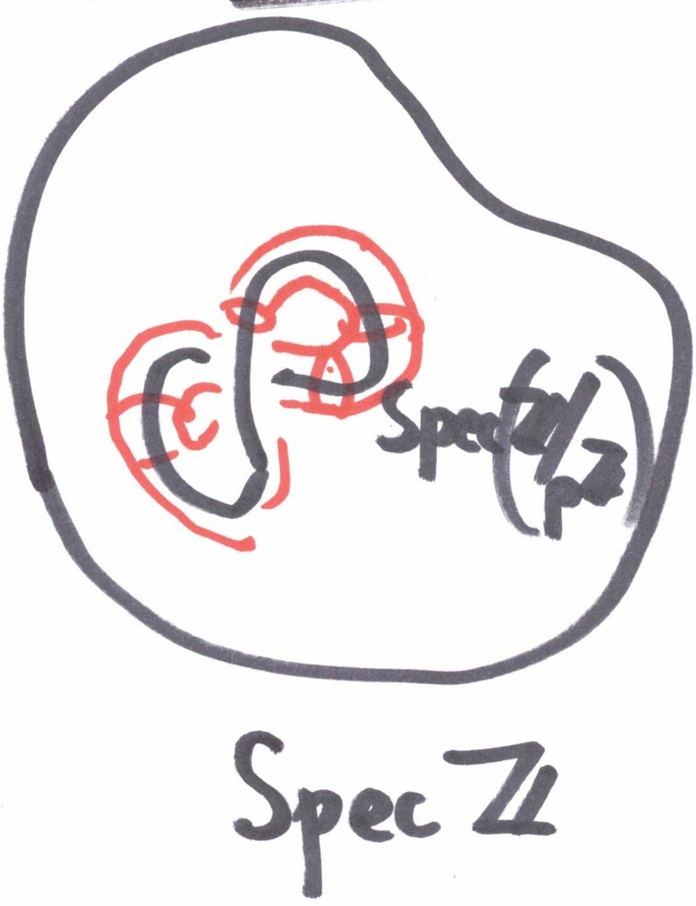
$$\begin{array}{c} \downarrow \\ \{i+1\} \end{array}$$

i.e.

$\mathbb{Z}[\frac{1}{p}]$ is like a (nonorientable) 3-manifold

with boundary $\mathbb{Q}_p$,

which is **like** a 2-manifold



← deleting tube around knot produces $\mathbb{Z}[\frac{1}{p}]$.

i.e. $\operatorname{Spec}\mathbb{Z}$ is like

$(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}])$ glued along

$(\operatorname{Spec}\mathbb{Q}_p)$ to $\operatorname{Spec}\mathbb{Z}_p$.

$\nearrow$ ∂(tube)

$\nearrow$ tube

$\partial(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}])$.

Garden of rings

3-diml rings / arithmetic objects

~~Spec~~ $\mathbb{Z}$, $\mathbb{Z}[\frac{1}{p}]$, $\mathbb{Z}[\sqrt{2}]$

$\mathbb{F}_q(t)$, proj. smooth curve our $\mathbb{F}_p$,

$\mathbb{Z}_p$.

2-diml objects

$\mathbb{Q}_p$, $\mathbb{F}_p((t))$,

proj. smooth curve our $\overline{\mathbb{F}_p}$

Number fields $\longleftrightarrow$ 3-manifolds

Automorphic forms $\longleftrightarrow$?

Aut. forms ($G \overset{e.g.}{=} SL_2$).

$$\mathbb{Z} \rightsquigarrow \left\{\begin{array}{c}\text{functions} \\ \text{on } G_{\mathbb{Z}} \backslash G_{\mathbb{R}}\end{array}\right\} = A_{\mathbb{Z}}$$

$$\mathbb{Z}\left[\tfrac{1}{p}\right] \rightsquigarrow \underbrace{\left\{\begin{array}{c}\text{functions} \\ \text{on} \\ G_{\mathbb{Z}[\frac{1}{p}]} \backslash G_{\mathbb{R}} \times G_{\mathbb{Q}_p}\end{array}\right\}}_{A_{\mathbb{Z}[\frac{1}{p}]}}$$

We would like

$$M \longrightarrow A_M$$

3-manifolds → vector spaces

which behaves similarly.

Nonexample: $M \longrightarrow H^*(M, \mathbb{C})$

behaves nothing like $\mathbb{Z} \rightsquigarrow A_{\mathbb{Z}}$.

e.g.

wrong functoriality. $\mathbb{Z} \rightarrow \mathbb{Z}[\sqrt{2}]$

"double cover"

$\mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}]$

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also:

$H^*(M \text{ union } N) = H^*(M) \oplus H^*(N)$.

$$A_{\mathbb{Z}\oplus\mathbb{Z}} = A_{\mathbb{Z}} \otimes A_{\mathbb{Z}}$$

- Number fields $\longleftrightarrow$ 3-manifolds

automorphic forms $\longleftrightarrow$ topological quantum field theory

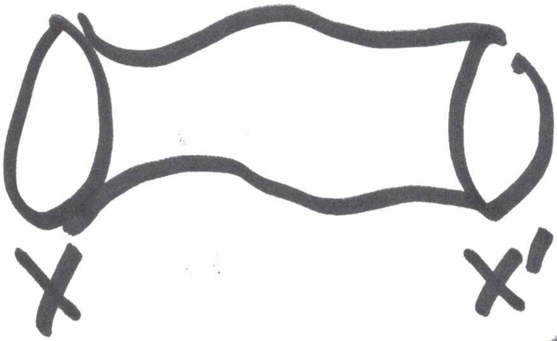
(Comes from work of Kapustin–Witten, 2006)

TQFT$_4$

functor

(3-manifolds, bordisms) → (vector spaces).

disjoint union ⟼ $\otimes$.



Atiyah.
TQFT, Section 2.