$TQFT_4$ (all in Atiyah §2)

3-manifolds M → vector spaces $A_M / \mathbb{C}$
disjoint union → ⊗
bordisms → linear maps
$A_M \to A_N$

$\Longrightarrow$ Invariants of 4-manifolds

4-manifold Z with boundary M → a vector in A_M

4-manifold Z without boundary → a complex number

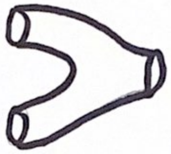
s.t.

→ ⟨left, right⟩
in A_M in A_M^*

Example (but of a $TQFT_2$) Dijkgraaf Witten

Fix a finite group G

→ ~~class functions on G~~ $[\mathbb{C}G]$ conjugacy-inv't



→ multiplication

genus g surface → $\frac{1}{|G|}$ (# of ways to write $e \in G$ as a product of g commutators)

Extended $TQFT_4$ (informal)

4-manifolds → $\mathbb{C}$
3-manifolds → vector spaces
2-manifolds → categories with Hom vector spaces
so:

→ object ∈ $\text{category}_S = A_S$

→ Hom(left, right)

Last time:

~~$\underbrace{\mathrm{Spec}\,\mathbb{Z}[\frac{1}{p}]}_{\text{3-manifold}}$ has boundary $\underbrace{\mathbb{Q}_p}_{\text{2-manifold}}$~~

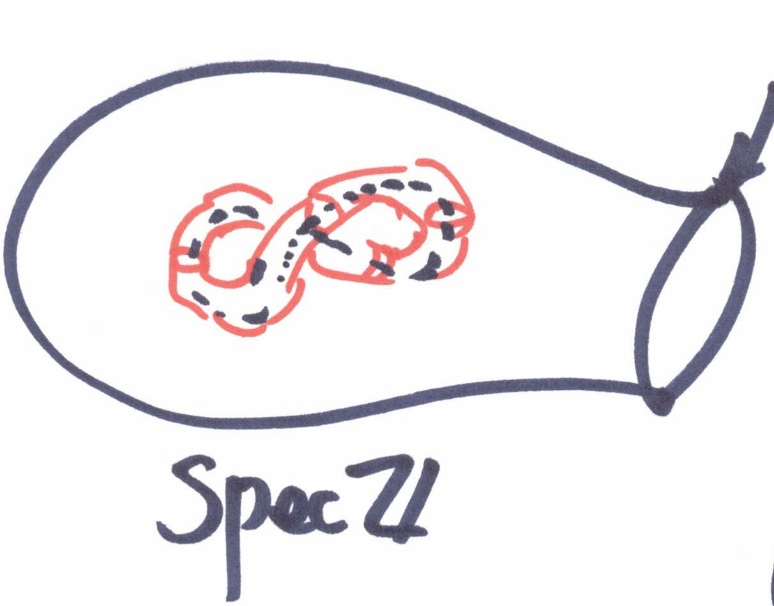
$\mathrm{Spec}\,\mathbb{Z}[\frac{1}{p}]$ has boundary

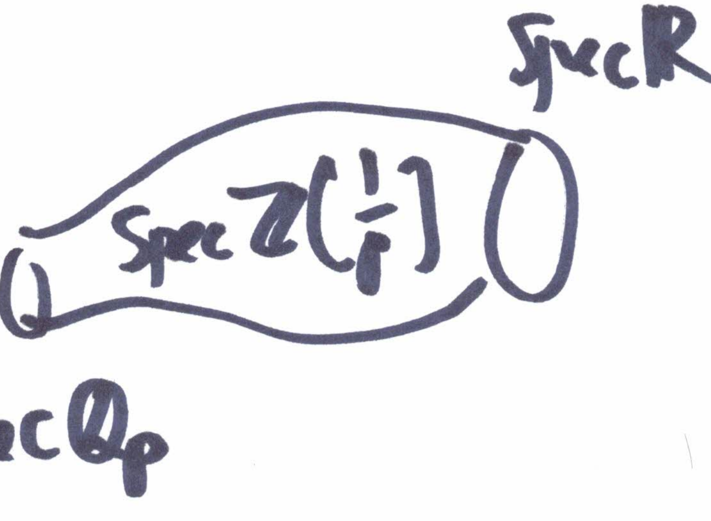
$\mathrm{Spec}\,\mathbb{R}$ · $\mathrm{Spec}\,\mathbb{Q}_p$

$\mathrm{Spec}\,\mathbb{Z}$ ——— $\mathrm{Spec}\,\mathbb{R}$

———

Spec ℝ





Aut. forms as extended TQFT$_4$.

3-dim'l

$\mathbb{Z} \longrightarrow A_{\mathbb{Z}} = \boxed{\text{functions on } G_{\mathbb{Z}} \backslash G_{\mathbb{R}}}$

$\mathbb{Z}[\frac{1}{p}] \longrightarrow A_{\mathbb{Z}[\frac{1}{p}]} = \boxed{\text{functions on } G_{\mathbb{Z}[\frac{1}{p}]} \backslash G_{\mathbb{Q}_p} \times G_{\mathbb{R}}}$

X smooth proj curve / $\mathbb{F}_p$ $\longrightarrow$ functions on G-bundles on X

$\mathbb{Z}_p \longrightarrow$???

2-dim'l $\mathbb{Q}_p \longrightarrow$ category of $G(\mathbb{Q}_p)$-repns

$\mathbb{R} \longrightarrow$ category of $G(\mathbb{R})$-repns

$\overline{X}$: proj smooth curve / $\overline{\mathbb{F}_p}$ $\longrightarrow$ category of sheaves on G-bundles on $\overline{X}$

Spec $\mathbb{Z}$ is obtained by gluing Spec $\mathbb{Z}_p$ to Spec $\mathbb{Z}[\frac{1}{p}]$ along Spec $\mathbb{Q}_p$.

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(A_{\mathbb{Z}_p}, A_{\mathbb{Z}[\frac{1}{p}]}\right) \overset{?}{=} A_{\mathbb{Z}}$$

$A_{\mathbb{Z}}$ = elements of $A_{\mathbb{Z}[\frac{1}{p}]}$ unramified at p.

$$= \mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\boxed{\text{functions on } G(\mathbb{Q}_p)/G(\mathbb{Z}_p)}, A_{\mathbb{Z}[\frac{1}{p}]}\right)$$

encodes $\nearrow$ Hecke at p

What is the Langlands <u>correspondence</u>?

Let's call $\mathcal{O} \longrightarrow A_{\mathcal{O}}$

$$\begin{tikzcd} \mathcal{O} \arrow[r] & A_{\mathcal{O}} \\ \text{arithmetic rings} \arrow[u] & \text{vec. space/category} \arrow[u] \end{tikzcd}$$

"arithmetic <u>field</u> theory."

X proj. smooth curve / $\mathbb{F}_p$
$G = GL_n$.

Langlands correspondence gives $\simeq$

$$\begin{tikzcd} \left\{\begin{array}{l}\text{cuspidal}\\ \text{functions}\\ \text{on } n\text{-dim'l}\\ \text{vector bundles}\end{array}\right\} \arrow[r, "\sim"] & \left\{\begin{array}{l}\text{functions}\\ \text{on } n\text{-dim'l}\\ \text{irreducible}\\ \text{Galois repns}\end{array}\right\} \\ X & \\ \circlearrowleft T_x \arrow[u] \arrow[r, leftrightarrow] & \circlearrowleft F_x \arrow[uu] \end{tikzcd}$$

This suggests the following viewpoint:

there's a *second* "arithmetic field theory", built out of Galois repns into $\check{G}$ = Langlands dual gp

$$B^{(\check{G})}$$

and an equivalence of arithmetic field theories

$$A^{(G)} \;\overset{\sim}{\rule{3em}{0.4pt}}\; B^{(\check{G})}$$

aut. form

$$\begin{tikzcd}
\text{aut. forms} \arrow[rr, bend left, "\text{Fourier, Rankin-Selberg, doubling, } \Theta \ldots"] \arrow[d, leftrightarrow] & & \mathbb{C} \\
\text{Galois repns} \arrow[urr, bend right, "L\text{-fn}"'] & &
\end{tikzcd}$$

$\exists$ zoo of matching invariants!

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e.g E elliptic curve $/\mathbb{Q}$

$$L(\mathrm{sym}^2 E, 1) = \boxed{\prod_p \frac{p^2}{(1-\frac{1}{p})\# E(\mathbb{F}_{p^2})}}$$

$$= (\text{rational}) \boxed{\pi \cdot \text{area}(E_{\mathbb{C}})}$$

aut. forms $\longrightarrow \mathbb{C}$

Galois repns $\longrightarrow \mathbb{C}$.

$\mathcal{O}$ = 3-diml ring of integers$/X$

numerical invt of Galois repns $\in B_{\mathcal{O}}^{(\check{G})}$.

numerical invt of aut forms $\in A_{\mathcal{O}}^{(G)}$.

given P here

it gives $\varphi \rightarrow \langle P, \varphi \rangle$.

To find matching invts, want matching elts of $A_{\mathcal{O}}$ & $B_{\mathcal{O}}$.

(Kapustin, 2010)

A boundary condition in a $TQFT_4$ (informal defn) is a consistent assignment

... to every 3-manifold M → distinguished vector in A_M.

... to each 2-manifold S → distinguished object in A_S

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want matching boundary conditions in $A^{(4)}$ & $B^{(4)}$.

Joint work with
Ben-Zvi, Sakellaridis.

informal summary:

1) G-variety Y gives boundary condition for $A^{(G)}$ & $B^{(G)}$

2) For suitable Y, this recoves all the familiar $\underbrace{\text{invts of aut. forms}}_{A}$ / $\underbrace{\text{L-function}}_{B}$

3) Propose a class of dual pairs $(G, Y) \longleftrightarrow (\check{G}, \check{Y})$ which give matching boundary condition