Lecture III

Automorphic G	Galois $G^\vee$
$\mathcal{A}_c(K_S, \chi_S)$ $\Downarrow$ f eigenform $\longrightarrow$	$\pi_1(X \setminus S) \longrightarrow G^\vee(\bar{\mathbb{Q}}_\ell)$
rigid auto data $\longrightarrow$	rigid loc sys. (Katz).

Designing Rigid auto. data.

1. Numerical Rigidity.

$\underbrace{\mathrm{Bun}_G(K_S)}(k)$

alg stack should have dim 0.

$$\dim \mathrm{Bun}_G(K_S) = 0$$

$$\Updownarrow$$

$$\sum_{x \in S} [G(\mathcal{O}_x) : K_x] = \underline{(1-g)\dim G}$$

relative dim $\geq 0.$

e.g. $K_x = I_x$

$$[G(\mathcal{O}_x) : I_x] = \dim\left(G(\mathcal{O}_x)/I_x\right)$$
$$= \dim(G/B)$$
$$= \# \Phi^+.$$

if $K_x \not\subset G(\mathcal{O}_x)$

$$[G(\mathcal{O}_x) : K_x] = \dim G(\mathcal{O}_x)/G(\mathcal{O}_x) \cap K_x$$
$$- \dim K_x / G(\mathcal{O}_x) \cap K_x$$

RHS ≥ 0 $\quad g = 0, 1.$

$g=0$: $\emptyset$

$g=1$: $K_x \sim G(\mathcal{O}_x)$. (very special)

Ex. $S = \{0, 1, \infty\} \subset \mathbb{P}^1$

K_x = parahoric subgp

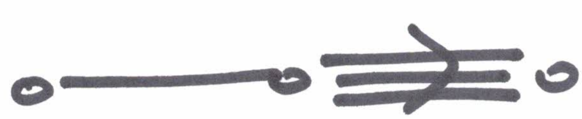
$$\sum_{x=0,1,\infty} [G(\mathcal{O}_x) : K_x] = \dim G$$

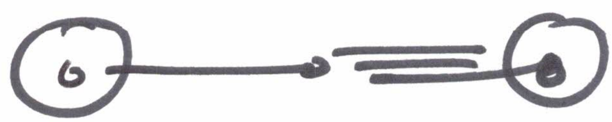
$K_x \twoheadrightarrow L_x$ = reductive quot. of K_x

$$[G(\mathcal{O}_x) : K_x] = \frac{\dim G - \dim L_x}{2}$$

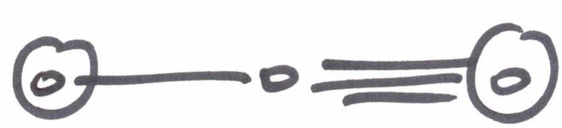
$$\sum_{x=0,1,\infty} \dim L_x = \dim G.$$

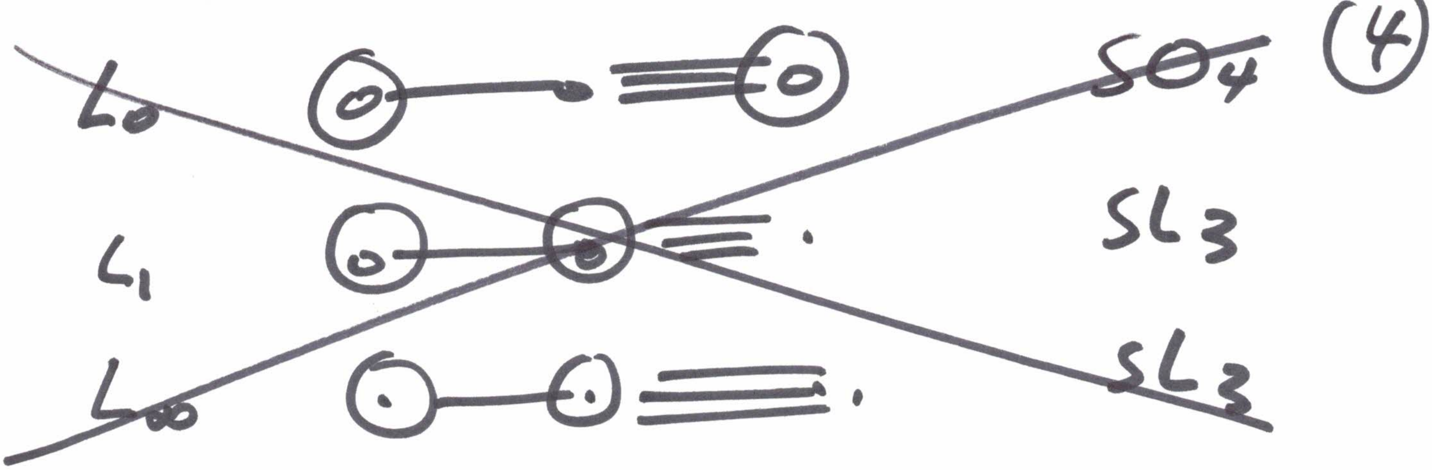
G_2.



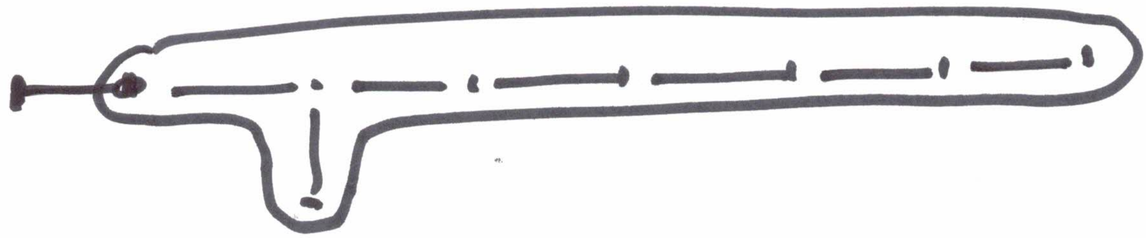
L_0  SO_4.

L_1 Iwahori T (2d)

L_∞  SO_4



$\underline{\underline{E_8}}$.



$L_0 = L_\infty$. $\qquad$ $\mathrm{Spin}(16)/\{\pm 1\}$ $\qquad$ 120×2

L_1 $\qquad$ Iwari $\twoheadrightarrow T$ $\qquad$ 8

$\overline{}$

248

χ_0. $\qquad$ $K_0 \twoheadrightarrow \cancel{\mathrm{Spin}}\, L_0(k)$

$$\downarrow$$

$$L_0(k)/\mathrm{Spin}(16)(k)$$

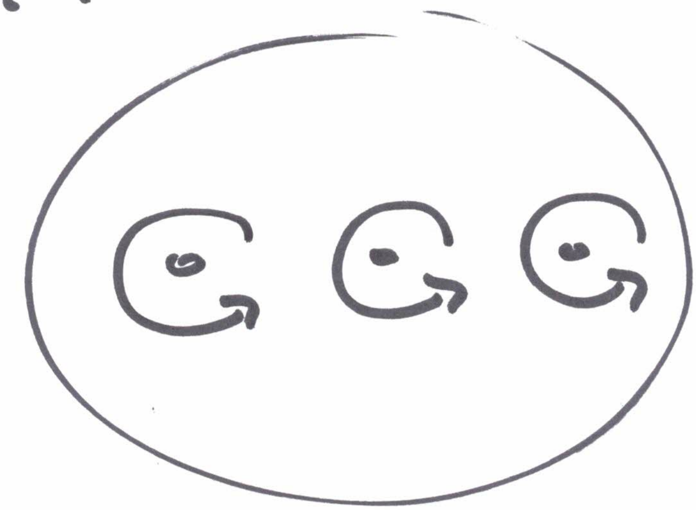
$$\|$$

$$\mathbb{Z}/2 \longrightarrow \{\pm 1\}$$

$\chi_\infty = 1$, $\chi_1 = 1$.

$\rightsquigarrow$ rigid auto. datum.

2. Matching auto data with local monodromy.



$$\begin{tikzcd}[column sep=small]
\text{Loc Sys} \arrow[d, mapsto] & & \\
\rho_x : & \mathrm{Gal}(\bar{F}_x/F_x) \arrow[r] & G^\vee(\bar{\mathbb{Q}}_\ell). \\
\text{inertia} & \mathrm{In}_x \arrow[u, phantom, "\cup" description] \arrow[ur] &
\end{tikzcd}$$

<u>Ex</u> $K_x = I_x \longrightarrow T(k) \xrightarrow{\ \chi\ } \bar{\mathbb{Q}}_\ell^\times$

$\Rightarrow \rho_x$ is tamely ramified.

$$\mathrm{In}_x \longrightarrow k_x^\times \xrightarrow[\text{given by } \chi.]{} \hat{T}(\bar{\mathbb{Q}}_\ell)$$

This is $\underline{\underline{(\rho_x|_{\mathrm{In}_x})^{ss}}}$.

<u>Ex</u> $K_x = I_x^+ \longrightarrow k \xrightarrow{\psi} \mathbb{C}^*$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto b + \frac{c}{t} \pmod t$$

$a, d \equiv 1 \ (t)$

$c \equiv 0 \ (t)$

$\rho_x : \mathrm{Gal}(\bar{F}_x / F_x) \longrightarrow \mathrm{Gl}_2(\bar{\mathbb{Q}}_\ell)$

wildly ramif.

$\mathrm{Sw}(\rho_x) = 1 = \frac{1}{2} + \frac{1}{2}.$

I_x^+

Moy-Prasad filtration on I_x.

indexed by $\frac{1}{h}\mathbb{Z}$ $\quad h =$ Cox number of G

($= 2$, $G = \mathrm{Sl}_2$)

$I_x = I_x(0) \supset I_x(\tfrac{1}{2}) = I_x^+$

$\supset I_x(1) \supset I_x(\tfrac{3}{2}) \supset \cdots$

If $K_x \subset P_x(r)$ $\in \mathbb{Q}$.

Then all slopes of ρ_x are $\leq r$.

depth

slopes.

<u>local Numerical condition</u>

(K_S, χ_S) rigid.

$\rightsquigarrow \rho : \pi_1 \longrightarrow G^\vee$

$$[G(O_x) : K_x] = \frac{1}{2} a(\mathrm{Ad}(\rho_x))$$

Artin conductor

<u>Ex.</u> (epipelagic auto. data)

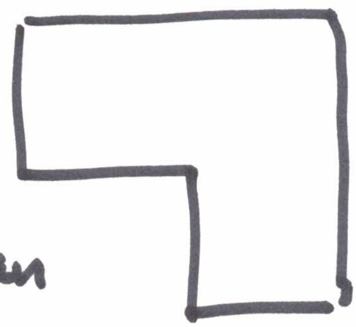
$S = \{0, \infty\}$.

$K_0 = P_0$ parahoric, $\chi_0 = 1$.

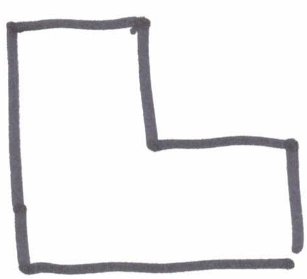
$K_\infty = P_\infty^+ \xrightarrow{*} k \xrightarrow{\psi} \mathbb{C}^\times$

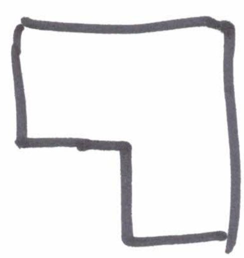
$G = Sp_{2n} = Sp(V)$

Siegel parabolic
||
stab. of a Lagrangian
$\subset V$


$= P_{Sieg}$

$$\begin{tikzcd} P_0 \arrow[d, two heads] & \subset & G(\mathcal{O}_0) \arrow[d, two heads] \\ P_{Sieg}^{opp} & \subset & G \end{tikzcd} \qquad \begin{tikzcd} P_\infty \arrow[d, two heads] & \subset & G(\mathcal{O}_\infty) \arrow[d, two heads] \\ P_{Sieg} & \subset & G \end{tikzcd}$$





Lag $= L \subset V$.

$$
\begin{tikzcd}
P_\infty^+ = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\mathcal{O}_\infty) \;\middle|\; \begin{array}{l} A, D \equiv I_n \pmod{\tau} \\ C \equiv 0 \pmod{\tau} \end{array} \right\} \arrow[d] \\
W \ni \left(B \bmod \tau, \ \dfrac{C}{\tau} \bmod \tau \right)
\end{tikzcd}
$$

$\cancel{W \cong \mathrm{Sym}^2}$

$\bar{A} \in GL(L)$.

$\bar{D} \in GL(L^\vee)$.

$\bar{B} : L^\vee \longrightarrow L \qquad \bar{B}^\vee = \bar{B}$

$\dfrac{C}{\tau} : L \longrightarrow L^\vee \qquad$ self-adj

$$W = \mathrm{Sym}^2(L) \oplus \mathrm{Sym}^2(L^\vee).$$

$\mathrm{Sym}^2(L) \ni S$, $\mathrm{Sym}^2(L^\vee) \ni T$

$$
\begin{array}{ccc}
W & & (X, Y) \\
\big\downarrow (S, T) & & \big\downarrow \\
k & \ni & \mathrm{Tr}(XT) + \mathrm{Tr}(YS)
\end{array}
$$

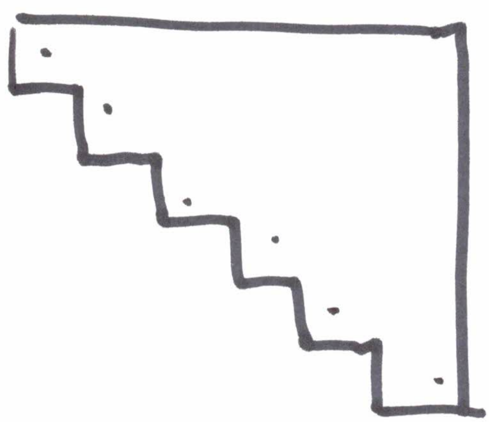
$$P^+_\infty \longrightarrow W \xrightarrow{(S,T)} k \xrightarrow{\psi} \mathbb{C}^*$$

For "stable" (S,T) we will get rigid auto datum.

stable means: $ST \in \mathrm{End}(L)$ has distinct $\neq 0$ eigenval in $\overline{k}$.

Epipelagic reps of $G(F_\infty)$
(Reeder - J.K. Yu)

Sp_{2n}.



equal sized block